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Logical letter status, a clue issue in the didactics of proof and proving

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In what follow, we will focus on the role of letters in university mathematics, considering both teachers' practises and undergraduates' difficulties.

In university mathematics education (UME)

- there is an increasing use of letters with various logical status.
- taking in account these various logical status is often a clue aspect in proof and proving.

Nevertheless, this remains often implicit in mathematical texts address to students, in particular when teachers consider it not *dangerous* (*Durand-Guerrier & Arsac, 2005*)

Predicate calculus
as
epistemological reference

Insufficiency of propositional calculus ; considering open statements (*Durand-Guerrier & Arsac 2005*).

First order logic in a semantic perspective : *Aristotle Frege, Russel, Wittgenstein, Tarski, Quine (Durand-Guerrier 2005, 2008).*

Dialectics between syntax and semantics, form and content, logical validity and truth in an interpretation (*Durand-Guerrier, 2005, 2008).*

Natural deduction (*Copi, 1954)***, a tool for analysing proof** (*Durand-Guerrier, 2008).*

Logical analyse of invalid proofs produced by prominent mathematicians (*Arsac, 2013).*

Quantification
and
logical status of letters

A crucial role of quantification in mathematical activity and reasoning

(Dubinsky & Yparaki, 2000, Selden & Selden, 1995, Epp 2004, Durand-Guerrier 2003, Chellougui 2009,)

related to

A crucial role of logical status of letters:

- ***singular object*** (a constant);
- ***generic element*** (an element considered only as a member of a given set or subset);
- ***free variable (place-holder***) that does not represent anything but is subject to assignation;
- ***bounded variable*** in the scope of a quantifier (a particular case of *mute letter*).

A classical example

The definition of the convergence of function

f has for limit $\mathbf{a}$ in 0 *iff*

$$\forall \boldsymbol{\varepsilon} > 0\ \exists \boldsymbol{\delta} > 0\quad [|\mathbf{x}| < \boldsymbol{\delta} \Rightarrow |\mathbf{f}(\mathbf{x}) - \mathbf{a}| < \boldsymbol{\varepsilon}]$$

$\boldsymbol{f}$ has for limit **a** in 0 *iff*

$$\forall \boldsymbol{\varepsilon} > 0 \ \exists \boldsymbol{\delta} > 0 \ \forall \mathbf{x} \in D_f$$

$$[|\mathbf{x}| < \boldsymbol{\delta} \Rightarrow |f(\mathbf{x}) - \mathbf{a}| < \boldsymbol{\varepsilon}]$$

$\boldsymbol{f}$ has a limit in 0 *iff*

$$\exists y\ \forall \boldsymbol{\varepsilon} > 0\ \exists \boldsymbol{\delta} > 0\ \forall \mathbf{x} \in D_f$$
$$[|\mathbf{x}| < \boldsymbol{\delta} \Rightarrow |f(\mathbf{x}) - \mathbf{y}| < \boldsymbol{\varepsilon}]$$

$\boldsymbol{f}$ has a limit in 0 *iff*

$$\exists y \in R\ \forall \boldsymbol{\varepsilon} \in R^{+*}\ \exists \boldsymbol{\delta} \in R^{+*}\ \forall \mathbf{x} \in D_f$$
$$[|\mathbf{x}| < \boldsymbol{\delta} \Rightarrow |f(\mathbf{x}) - \mathbf{y}| < \boldsymbol{\varepsilon}]$$

Logical status of letters and Natural deduction

Natural deduction enlightens the role of ***introduction and elimination of connectors and quantifiers***, and may contribute to ***checking mathematical proofs.***

I present briefly ***the rule for quantifiers*** of Copi's system (from Copi, 1965) and give ***an example of proof's checking***.

Universal Instantiation : elimination of the universal quantifier From a **bounded variable** to **a generic element**

$$\frac{\forall x F(x)}{F(a)}$$

a is an individual constant and *F*(*a*) results from *F*(*x*) by replacing all free occurrences of *x* in *F*(*x*) by *a.*

"What is true for all, is true for any"

Universal Generalisation : Introduction of the universal quantifier

From **a generic element** to **a bounded variable**

$$\frac{F(a)}{\forall x F(x)}$$

First restriction rule:

a denotes any arbitrarily selected element, without any assumption other than its belonging to the considered domain.

Existential Generalisation : Introduction of the existential *quantifier*

From **a generic element** to **a bounded variable**

$$\frac{F(a)}{\exists x F(x)}$$

a is any individual symbol.

“existence can be inferred from a true instance”

Existential Instantiation : elimination of the existential quantifier

$$\frac{\exists x F(x)}{F(w)}$$

Second restriction rule

It is necessary to be aware of the interpretation of *w*: *w* is "any individual constant which has had no prior occurrence in that context and is used to denote the individual, or one of the individuals, whose existence has been asserted by the existential quantification.". (Copi, 1965,. p.52).

This fourth rule is the more delicate to use, and necessitates a global control of the proof or of the argument. Indeed, as we have shown in Durand-Guerrier et Arsac (2005), a control step by step is not enough to track possible invalid deduction.

Copi uses *x, y, z* exclusively for bounded variable, and *a, b, c* for generic elements or constant.

“Ce système offre l’avantage de proposer des demonstrations qui restent au plus près de l’aspect familiers des syllogismes. Cette présentation correspond à la volonté de formaliser et d’axiomatiser en ne rompant pas avec la rationalité discursive naturelle.” (Hottois, 1989, p. 100).

Using natural deduction
for
checking the validity of a proof.

Theorem:

Given two functions f and g defined in a subset A of the set of real numbers, and a an adherent element of A, if $f(t)$ and $g(t)$ have h and k, respectively, for limits as t tends to a remaining in A, then $f + g$ has $h + k$ for limit in a.

A proof of the previous theorem:

"By hypothesis,

for all $\varepsilon > 0$, there exists $\eta > 0$ such that $t \in A$ and $|t - a| \leq \eta$ imply

$$|f(t) - h| \leq \varepsilon$$

and

$$|g(t) - k| \leq \varepsilon ;$$

thus, we have

$$|f(t) + g(t) - (h + k)|$$
$$= |f(t) - h + g(t) - k|$$
$$\leq |f(t) - h| + |g(t) - k| \leq 2\varepsilon."$$

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Premises for the first part of the proof

P_1 - For all $\varepsilon > 0$, there exists $\eta > 0$ such that $t \in A$ and $|t - a| \leq \eta$ imply $|f(t) - h| \leq \varepsilon$

P_2 - For all $\varepsilon > 0$, there exists $\eta > 0$ such that $t \in A$ and $|t - a| \leq \eta$ imply $|g(t) - k| \leq \varepsilon$;

First part of the proof

Let ε_0 be a strictly positive number.

UI on (P_1) provides an existential statement (S_1); idem for UI on (P_2), that provides (S_2).

By EI on (S_1), we introduce η_1 such that $t \in A$ and $|t - a| \leq \eta_1$ imply $|f(t) - k| \leq \varepsilon_0$.

By EI on (S_2), we introduces η_2 such that $t \in A$ $|t - a| \leq \eta_2$ imply $|g(t) - k| \leq \varepsilon_0$.

To go on, we will eliminate the two implications with an auxiliary premise (hypothesis)

[**H**: $|t - a| \leq \eta_1$ and $|t - a| \leq \eta_2$

$|t - a| \leq \eta_1$ H1 - *separation on H*

$|f(t) - k| \leq \varepsilon_0$ C1-*Modus Ponens on H1*

$|t - a| \leq \eta_2$ H2 – *separation on H*

$|f(t) - k| \leq \varepsilon_0$ C2 - *Modus Ponens on H2*

$|f(t) - k| \leq \varepsilon_0$ and $|g(t) - k| \leq \varepsilon_0$ **C**]

At this point, there is no way to go on; indeed, a **mathematical step is missing** in the proof.

Missing mathematical step:

Let $\eta_3 = \min(\eta_1, \eta_2)$

$|t - a| \leq \eta_3 \Rightarrow$

$|t - a| \leq \eta_1$ and $|t - a| \leq \eta_2$ (S_3)

A mathematical argument is explicitly used:

The set of real numbers is totally ordered, so it is always possible to consider the minimum of a pair of elements.

Modification of the auxiliary hypothesis

[**H:** $|\boldsymbol{t} - \boldsymbol{a}| \leq \boldsymbol{\eta_3}$

$|t - a| \leq \eta_1$ and $|t - a| \leq \eta_2$ H_1 – *MP* on S_3

$|t - a| \leq \eta_1$ H_2 - *separation on H_1*

$|f(t) - k| \leq \varepsilon_0$ C_1-*Modus Ponens on H_2*

$|t - a| \leq \eta_2$ H_3 – *separation on H2*

$|f(t) - k| \leq \varepsilon_0$ C_2 - *Modus Ponens on H_3*

$|\boldsymbol{f}(\boldsymbol{t}) - \boldsymbol{k}| \leq \boldsymbol{\varepsilon_0}$ **&** $|\boldsymbol{g}(\boldsymbol{t}) - \boldsymbol{k}| \leq \boldsymbol{\varepsilon_0}$ C- C_1 & C_2]

To achieve the (first part of) the proof:

Introduction of implication

$|t - a| \leq \eta_3 \Rightarrow |f(t) - k| \leq \varepsilon_0$ & $|g(t) - k| \leq \varepsilon_0$

Introduction of quantifiers

First, the ***existential*** one; then the ***universal*** one:

$\forall \boldsymbol{\varepsilon} > 0, \exists \boldsymbol{\eta} > 0$ $(|t - a| \leq \boldsymbol{\eta}$

$\Rightarrow |f(t) - h| \leq \varepsilon$ and $|g(t) - k| \leq \boldsymbol{\varepsilon}$

A proof of the previous theorem:

"**By hypothesis**,

for all $\varepsilon > 0$, there exists $\eta > 0$ such that $t \in A$ and $|t - a| \leq \eta$ imply

$$|f(t) - h| \leq \varepsilon$$

and

$$|g(t) - k| \leq \varepsilon;$$

thus, we have

$$|f(t) + g(t) - (h + k)| = |f(t) - h + g(t) - k| \leq |f(t) - h| + |g(t) - k| \leq 2\varepsilon."$$

The proof is out of a french textbook for first year University students.

The author *seems* to use the invalid inference rule:

For all x there exist y such that F(x, y) and for all x there exists y so that G(x, y) ; hence for all x there exists y such that (F(x,y) and G(x,y)).

In the example, given the premises are true, the conclusion is true.

In other cases, it is possible to have **the premises true** and **the conclusion false.**

So, given the premise are true, the truth value of the conclusion is depending on the context.

This is hidden in the proof of the textbook where the mathematical argument justifying that the conclusion is true is absent.

However, as university teachers know, applying this invalid inference in non appropriate context is a frequent error among undergraduates.

We have here typically ***an obstacle*** with not only ***cognitive*** origin, but also ***epistemological*** and ***didactical*** ones (Brousseau, 1997).

Concerning students, there are already research-based evidence that uncertainty on the logical status of letter provoke difficulties in proof and proving in mathematics, in particular when statements with alternance of quantifiers are involved (Dubinski & Yparaki, Selden and Selden, 1995, Durand-Guerrier et Arsac, 2005, Chellougui 2009, Barrier 2009, Njomgang Ngansop 2013).

Floating use of letters in mathematical texts

An example In Buck and Buck, 1965 p. 68

For another example, let $f(x) = 1/x$. We shall show that f is continuous on the open interval $0 < x < 1$ but is not uniformly continuous there. We first write

$$|f(x) - f(x_0)| = \left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x_0 - x|}{xx_0}$$

To prove continuity at x_0, which may be any point with $0 < x_0 < 1$, we wish to make $|f(x) - f(x_0)|$ small by controlling $|x - x_0|$. If we decide to consider only numbers δ obeying $\delta < x_0/2$, then any point x such that $|x - x_0| < \delta$ must also satisfy $x > x_0/2$, and $xx_0 > x_0^2/2$. Thus, for such x, $|f(x) - f(x_0)| < \delta/xx_0 < 2\delta/x_0^2$. Given $\epsilon > 0$, we can ensure that $|f(x) - f(x_0)| < \epsilon$ by taking δ so that $\delta \leq (x_0^2/2)\epsilon$. Thus, f is continuous at each point x_0 with $0 < x_0 < 1$. If f were uniformly continuous there, then a number $\delta > 0$ could be so chosen that $|f(x) - f(x')| < 1$ for every pair of points x and x' between 0 and 1 with $|x - x'| \leq \delta$. To show that this is not the case, we consider the special pairs, $x = 1/n$ and $x' = \delta + 1/n$. For these, we have $|x - x'| = \delta$ and

$$|f(x) - f(x')| = \left|n - \frac{1}{\delta + 1/n}\right| = \frac{n\delta}{\delta + 1/n}$$

No matter how small δ is, n can be chosen so that this difference is larger than 1; for example, any n bigger than both $1/\delta$ and 3 will suffice.

We focus on the logical status of the letter x.

Line 1 – Letter *x* is used as ***a free variable*** in the definition of function *f*.

Line 2 – Letter *x* intervenes in an unusual notation for the open interval]0;1[. (0 < x <1);

Line 4 - Although *x* and x_0 have not been explicitly introduced, they seem to have the status of ***generic elements***.

Line 5 - The status of ***generic element for x_0*** is confirmed but nothing is said concerning *x*.

Lines 7 and 8 - Letter *x* appears in a sentence that could be formalized as a universal conditional statement in which *x* would be ***a bounded variable.***

Line 9 - In the first sentence, letter *x* denotes a generic element satisfying "$|x - x_0| < \delta$" (implicit Universal Instantiation)

Lines 9 and 10 - In the sentence

"Given ε > 0, we can ensure that "$|f(x) - f(x_0| < \varepsilon$"

by taking δ so that $\delta \leq (x_0^2/2)\varepsilon$",

there is no indication that the letter *x* is a bounded variable in the scope of an implicit universal quantifier, corresponding to a Universal Generalization on *x*.

Consequently, the difference of logical status between the ***generic element*** $\boldsymbol{x_0}$, and the ***bounded variable x*** remains implicit.

Line 13 - The universal quantification on both *x* and *x*' is explicit. This appears to express what would occur if *f* were uniformly continuous.

Of course, the authors know that in this case making explicit the quantification is crucial. But what for students ?

Line 15 – Letters *x* and *x*' are used to label the pairs (1/*n*; δ+1/*n*) depending on *n*, a generic natural number that has not been introduced, without any indication of this dependence.

Such a *floating use of letters* appears also in some French textbooks. In the above case, it is particularly problematic because it might prevent the expected contribution of formalism to the conceptual clarification on a delicate mathematical concept.

An example of students' difficulties with the use of letters

We present some empirical results on a less complex statement, out of a questionnaire aiming to test students understanding of real numbers, submitted to fresh university students in France in Montpellier in collaboration with Laurent Vivier in March 2015.

The following question was the 7th one among 9.

Give all the real numbers that satisfy the following property

$$\forall k \in \mathbb{N}\ |a-1| \leq\ 10^{-k}$$

The questionnaire was submitted to two groups of students in first year university having followed the same course in analysis in first semester.

Group A1 were oriented to physics

Group A3 had been selected to follow a specific program preparing competitive exam for Engineering schools.

Brief a priori analysis

Main justifications expected for the correct answer :

- 1 is the only real number satisfying the property.
- This is the characteristic property of equality for real numbers.
- The sequence of general term 10^{-k} converge to 0 and an absolute value is positive.
- The two sequences are adjacent and converge toward 1.

N.B. Quantification matters are absorbed by the use of limit of sequence.

A (sometimes) incomplete argument

The given statement is equivalent to

$$\forall k \in \mathbb{N}(1 - 10^{-k} \leq a \leq 1 + 10^{-k})$$

Samples of students answers (1)

Elimination of the universal quantifier without changing the name of the bounded variable

$$1 - \ 10^{-k} \leq \ a \leq 1 + 10^{-k}$$

Transformation of the inequality as an interval

$$a \ \in [1 - \ 10^{-k}, 1 + 10^{-k} \ ]$$

In many cases, there is no explicit conclusion.

In some cases, students conclude that the real numbers which satisfy the property are the elements of the interval:

A3-11 “This property leads to the following set

$$I = [1 - \ 10^{-k}, 1 + 10^{-k} \ ]$$

Hence this property is true for all a$\in$I“

Samples of students' answers (2)

In the previous case, the student seems to consider that letter *k* refers to a constant. It is also the case for A1-26 or A3-26.

In other answers, letter *k* seems to be considered as a parameter:

A1-7 : solving an inequality in *k* by squaring.

A3-14 : solving an inequality in *k* using logarithm

A3-61: solving an inequality in *k* using square root

A student seems to look for a set E_k such that $a \in E_k$ *iff* $|a-1| \leq 10^{-k}$

A3-48 : $|a-1| \leq 10^{-k}$. The values satisfaying the property are depending on *k*; it is not possible to give all of them.

Samples of students answers (3)

Some students propose answers in which the logical status of letter *k* is changing in the same sentence:

A3-32 : the real numbers that satisfy the property are those equal to $\mathbf{10^{-k}+1}$ because for all **k** we have $|a-1| < 10^{-k}$.

(incorrect conclusion)

A3-55: a = 1 is the only number for which we can assert that it satisfies the inequality for all **k** because $\mathbf{10^{-k}}$ might be infinitively small but will be always positive.

(correct conclusion)

Conclusion

The empirical results support the claim that some fresh university students have to cope with uncertainty in logical status of letters.

Moreover, the study of I. Ben Kilani (Tunisia) and J. Njomgang Ngansop (Cameroon) show that the difficulties are still more important in multilingual context where the language of education is not the mother tongue.

We hypothesize that this is likely to reinforce difficulties in the various mathematical fields they study. In France, students arriving at university on the one hand are not prepared to deal with formalised statement, on the other hand most often do not master the mathematical content at stake.

These results lead to questioning the ordinary practices of University teachers concerning the use of letters in proof and proving.

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