PAT 2023 - Proof assistants for teaching proof and proving
19 - 23 June 2023

# *Logical issues in the teaching and learning of proof and proving*

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Monday June 19, 9h-10h30, V. Durand-Guerrier

*A semantic perspective on truth and validity in Mathematics. Contributions to didactics analysis.*

Preliminary – Proofs analysis

Tuesday June 20, 16h-17h, A. Meyer

*Syntax, semantics and proof in CS education*

Wednesday June, 21, 11h-12h30, V. Durand-Guerrier & A. Meyer

*Logical letter status, a clue issue in the didactics of proof and proving*

Logical issues
in the teaching and learning of proof and proving

# Session 1 – June 19, 2023

# *A semantic perspective on truth and validity in Mathematics*
# *Contributions to didactics analysis*

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The truth value of the statement « *The sum of the angles of a triangle is equal to 180°* » depends on the considered interpretation (theory) ; it is true in Euclidean Geometry, false in Hyperbolic Geometry.

The equivalence between a conditional statement and it contrapositive is logically valid.

The formula

$$\left[\forall \boldsymbol{x}\left(\boldsymbol{P}(\boldsymbol{x}) \Rightarrow \boldsymbol{Q}(\boldsymbol{x})\right)\right] \Leftrightarrow \left[\forall \boldsymbol{x}\left(\neg \boldsymbol{Q}(\boldsymbol{x}) \Rightarrow \neg \boldsymbol{P}(\boldsymbol{x})\right)\right]$$

is interpreted by a true statement whatever the interpretation, given that the universe is non empty (e.g. Quine, 1950).

*Our claim* : The distinction between *truth in an interpretation* and *logical validity* is crucial for the learning of proof and proving in mathematics (Durand-Guerrier, 2008)

There are research based evidence that generally students and teachers are not familiar enough with first order logic for approaching fruitfully these questions.

It can be observed all along schooling, and particularly at the beginning of tertiary level, that formalised mathematical language is a very resistant obstacle for many students.
A didactic study of these difficulties enlightens the fact that the meaning seems to be lost, so that the potentiality of formalisation and syntactical methods for learning are vanishing.
To articulate form and content, syntax and semantics, appears thus to be very important in mathematics teaching and learning at every level, and those opposite with a common idea that mastering syntax would be the priority.

As Sinaceur (1991) states for mathematicians, we consider that Tarski's construction of a definition of truth materially adequate and formally correct, and his project of bridging formal system and reality (including mathematical theories) offer an effective epistemology in mathematic education, opening paths for analysing mathematic activity and reconsidering a traditional assumption in mathematic education that there is a great distance between common sense and mathematical logic.

In mathematics education, many authors emphasise the fact that once students are convinced that something is true, they do not understand the  need for proof.

In this presentation, we will present arguments, both epistemological and didactical, that taking in account the distinction made in logic between truth and validity on the one hand, syntax and semantics on the other hand could contribute to better understand mathematical practices, in particular for what concerns proof and proving.

## *First order logic as an epistemological reference for didactic analysis of proof and proving in mathematics*

An epistemological enquiry considering philosophers from Aristotle to Quine, including Frege, Russell, Wittgenstein and Tarski, makes clear that the need to determine the appropriate distance between common sense and logic is at the very core of their theoretical considerations (Durand-Guerrier 2008).

The main categories that emerge are *truth* and *validity* on the one hand, *syntax* and semantics on the other.

Obviously, syntax and semantics are more general categories, however syntactics and semantics methods concern both truth and validity.

It is generally admitted that formal logic starts with Aristotle's Syllogism theory as presented in the first Analytics.

As Lukasiewicz put it, for Aristotle pure logic is what remains when material has been taken away (Lukasiewicz, 1951, 1972, p.22).

To build his system, Aristotle in *On the interpretation*

- extracted formal statements from ordinary language sentences
- emphasise the distinction between *contradiction* that applies to pair of statements such that their respective forms guaranty that their truth values are necessary different (e.g. Every A is B / Some A is not B) and *contrariety* a more radical opposition that applies to pair of statements such that their respective forms allow that, in some cases, both statements might be false (e.g. Every A is B / No A is B).

In the First Analysis, Aristotle offers a precise definition of a syllogism:

A conditional statement with two premises and a conclusion that respect a set of precise constitutive rules.

- The premises and the conclusion are quantified propositions with a subject term and a predicate term (an attribute);
- The medium term occurs twice in the premises ; it does not occur in the conclusion.
- The position of the medium term determines four figures.

In the first figure, the medium term is once subject and once predicate.

“I call *perfect* syllogism one who does not need anything else just what is assumed in the premises so that the need for the conclusion is obvious.

”[I call] *imperfect* syllogism, one that requires one or more things, which indeed necessarily result from the terms laid, but are not explicitly set out in the premises. ” (pp 4-5)

The perfect syllogisms are the conclusive syllogisms of the first figure characterized by the fact that the medium term is successively subject and predicate.

The perfect syllogisms of the first figure

If every *B* is *A*, and every *C* is *B*, then every *C* is *A*.

If no *B* is *A* and every *C* is *B*, then no *C* is *A*

If every *B* is *A* and some *Cs* are *B* , then some *Cs* are *A*

If no B is *A* and some *Cs* are *B*, then some *Cs* are not *A*

Example of interpretation of the universal syllogism of the first figure :

If every <u>man</u> is mortal and if every Greek is <u>man</u>, then every Greek is mortal.

For each of the syllogisms of the first figure that are not perfect, Aristotle gives an example showing that two different conclusions can be drawn under the terms chosen so that none of the two conclusions is necessary.

After having established the results for the first figure, Aristotle defines the other figures that consists in changing the position of the term that is present twice (medium term).

When the obtained syllogism is conclusive, he proves it by reducing it to a perfect syllogism of the first figure using the two conversion rules below:

- R1: some *A* is *B* may be replaced by some *B* is *A*
- R2: no *A* is *B* can be replaced by no *B* is *A*

For inconclusive syllogisms, he proceeds as he did for the first figure.

Conclusive syllogisms of Aristotle are logical laws that guarantee the validity of inference, to the extent that they give rise to a true statement whatever the interpretation of the terms *A*, *B* and *C*. Here we find the idea of universal validity as defined by Quine (1950).

To show that a syllogism is not a logical law (not conclusive), simply find an interpretation in which the premises are true and the conclusion is false.

Aristotle explicitly distinguishes between *necessary* truths obtained as conclusion of a syllogism concluding with true premises, and *de facto* truths in agreement with the facts.

In doing so, Aristotle combines *syntactic* and *semantic* methods and highlights the distinction between logical validity and truth in an interpretation as developed in first order logic in following the work of Wittgenstein (1921) and Tarski (1936).

Even if the system developed by Aristotle is not sufficient for the need of mathematical reasoning, he has developed logical concepts that remain essential in modern logic. As such, he is acknowledged as a precursor by many authors, as for example Largeaut (1972) who considers that Aristotle's use of both semantics interpretation and formal derivation attests of his "genial lucidity".

In a didactic perspective, this system allows a first encounter with the fundamental concepts of logic semantics that play a central role in proof and proving in mathematics. From 1994 to 2007, we have developed and implemented every year an optional university module "Logical statements and mathematical reasoning - epistemological aspects and didactic analysis". This experience showed that the logic of Aristotle provides a relevant context to address fundamental concepts of contemporary logic with mathematics students. (Durand-Guerrier 2016)

Among the philosophers who have put a strong emphasis on the distinction between truth and validity are Wittgenstein (1921) in the Tractatus Philosophicus where he develop a semantic version of the Propositional Calculus, and Tarski who provides a semantic recursive definition of truth for quantified logic and developed the methodology of deductive sciences, a first version of the elementary model theory (Tarski (1936, 1944, 1955).

Both authors are relying on a conception of truth similar to the one by Aristotle, and, as Aristotle did, emphasised the distinction between *truth in an interpretation* and *logical validity*.

Wittgenstein (1921) developed a semantic version of the Propositional Calculus, where the distinction between truth in an interpretation and logical validity is clarified.

4.062. (…) For a proposition is true if we use it to say that things stand in a certain way, and they do; (…)

6.113. It is the peculiar mark of logical propositions that one can recognize that they are true from the symbol alone, and this fact contains in itself the whole philosophy of logic. And so too it is a very important fact that the truth or falsity of non-logical propositions cannot be recognized from the propositions alone.

6.1263. (...) It is clear from the start that a logical proof of a proposition that has sense and a proof in logic must be two entirely different things.

6.1264. A proposition that has sense states something, which is shown by its proof to be so. In logic every proposition is the form of a proof. Every proposition of logic is a modus ponens represented in signs. (And one cannot express the modus ponens by means of a proposition.)

The English translations are those of the ebook prepared by M. Stapleton in the frame of the Project Gutenberg ebook: :https://archive.org/stream/tractatuslogicop05740gut/tloph10.txt

*Semantics*

*and*

*Methodology of deductive sciences*

Tarski (1936, 1944, 1955)

# The concept of truth in formalised languages

The purpose of Tarski was ”to construct a definition of the expression “true proposition” that would be materially adequate and formally correct” (Tarski 1933a, b, 1972, 1983, p. 159).

“the truth of a proposition lies in its agreement (or correspondence) with reality; or a proposition is true if it designates an existent state of things (Tarski 1944a, b, 1974, pp. 270–271)

# The concept of truth in formalised languages

To elaborate a recursive construction of truth for propositions,Tarski introduced the more general concept of “satisfaction of a propositional function (a predicate) by such or such objects” taking into account the fact that “complex propositions are not aggregates of propositions, but obtained from propositional functions” (Tarski 1933a, b, 1972, 1983, p. 193).

# The concept of satisfaction of a propositional function

For all a, a satisfies the propositional function *"* x is white*"* *if and only if* a is white.

For all a and b, a and b satisfy the propositional function *"x sees y" if and only if a* see *b.*

Link with the defintion of truth

By substituting *a* to *x* in "*x is white"*, one gets a proposition "*a is white"*

According to the general definition of truth, the proposition « *a is white*» is true *if and only if* a is white.

# An extension of logical connectors

*Negation*

"the negation of propositional function F(x) is satisfied exactly by those elements that do not satisfy F(x)"

In classical First order logic, "F(x) or non F(x)" is true in any interpretation

It is the extension of the *excluded middle principle* to propositional functions.

In the set of natural numbers, "to be an odd number" is the negation of the property "to be an even number".

"a natural number is either even, or not even (odd)"

# An extension of logical connectors

*Implication*

"In a given domain, an ***open implication*** "$P(x) \Rightarrow Q(x)$" is satisfied by those elements satisfying both « $P(x)$ » and « $Q(x)$ » (the examples), et by those that do not satisfy « $P(x)$ » ; it not satisfied only by those elements that satisfy « $P(x)$ » and that do not satisfy « $Q(x)$ » (the counter-examples) (recursive definition of satisfaction)

*Example:* determine all the natural numbers between 1 and 20 that satisfy the property:

"***if*** a natural number is even, ***then*** its successor is prime"

*Most frequent answer* : 2, 4, 6, 10, 12, 16, 18

*Correct answer*: 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19

# Model of the formula

A formalised language L, a syntax, well-formed statemens (formulae) : *F, G, H*…

An interpretative structure Σ (a domain of reality, a mathematical theory, a local theory)

Σ Is a *model of a* formula *F* of L if and only if the interpretation of *F* in Σ is a true statement.

- *F* : $\forall x \, \forall y \, (S(x,y) \wedge S(y,x) \Leftrightarrow x=y)$
- Σ : the ordered set of real numbers
- *S* → relation '*to be less or equal to* '

*F* → « the relation '*to be less or equal to* ' is antisymmetric »

# On the concept of logical consequence

This allows Tarski to define the fundamental notion of "logical consequence in a semantic point of view":

a formula G follows logically from a formula F *if and only if* every model for F is a model for G (Tarski 1936c, 1972).

This then means that the formula "F $\Longrightarrow$ G" is true for every interpretation of F and G in every nonempty interpretative structure (Quine 1950).

Example:

"Q($x$)" is a logical consequence of "P($x$) ∧ (P($x$) $\Rightarrow$ Q($x$))".

# On the concept of logical consequence

- « $\neg(\forall x(P(x)))$ » *follow logically of* « $\exists x(\neg P(x))$ » (1)
- « $\exists x(\neg P(x))$ *follow logically of* « $\neg(\forall x(P(x)))$ » (2)
- « q(x) » *follow logically of* « $p(x) \wedge (p(x) \Rightarrow q(x))$ » (3)
- « $\neg p(x)$ » *follow logically of* « $\neg q(x) \wedge (p(x) \Rightarrow q(x))$ » (4)

- *(1) et (2) correspond at the interdefinability of the two quantifiers of predicate calculus and is associated with the counterexample rule in mathematics.*
- *(3) corresponds to Modus Ponens*
- *(4) correspond to Modus Tollens*

Tarski (1936a, b,1969) established the two following important results:

“Every theorem of a given deductive theory is satisfied by any model of the axiomatic system of this theory; moreover at every theorem one can associate a general logical statement logically provable that establishes that the considered theorem is satisfied in any model of this type (…).” (Deduction theorem).

“All the theorems proved from a given axiomatic system remain valid for any interpretation of the system.”

These two fundamental theorems illustrate the relationship between semantics and syntax and lead to a very important method of proving that a statement is not a logical consequence of the axioms of a theory.

# Proof by interpretation

Proving that a given statement in a theory is not a logical consequence of the associated axiomatic system, consists in providing a model of the theory that is not a model of the formula associated with the statement in question.

Doing this, Tarski clarifies the distinction between *truth in an interpretation*, and *truth as a logical consequence of an axiomatic system*, which recovers Aristotle's original distinction between *necessary truth* and *de facto truth*.

# Proofs by interpretation

$\Sigma$ a formal axiomatic system
(without reference to objects)
A formula $\Phi$ associated to F

**A deductive theory** T
Axioms
A true statement F

A **Model** M of $\Sigma$
G the interpretation of $\Phi$
G is a false statement

F is not a logical consequence of the axioms in T

# An example – Tilling a grid

This example is presented in Durand-Guerrier & al. 2019.

The problem is the following:

Given a rectangular grid (with integral dimensions), is it possible to tile it with dominoes (1x2 rectangles)?

***Theorem***. A rectangular grid can be tiled by dominoes *iff* its area is even.

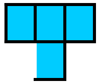
A frequent (incorrect) proof of the above theorem given by students is the following.

A grid can be tiled by dominos *if and only if* its area is 2k where k is the number of dominoes, which means that the area of the grid is even.

The stated theorem is correct but the proof is not.

It is sometimes difficult to invalidate an incorrect proof of a true statement. In order to do so, one can notice that the fact that the grid is rectangular was not used in the proof.

So, if this proof was correct, it could be used for any shape consisting of an even number of squares.

It is easy to see that the shape 

can not be tiled by dominoes but has an even area.

In other words, the set of grids of arbitrary shapes is a model of the theory used in the proof above. But in this model, the theorem becomes false.

Hence, the initial proof is invalid (because otherwise it could be transported into the new model).

# A fruitful theory for analysing mathematical activity

As Sinaceur (1991) states for mathematicians, we consider that Tarski's construction of a definition of truth *materially adequate* and *formally correct*, and his project of bridging *formal system* and *reality* (including mathematical theories) offer an effective epistemology in mathematic education, opening paths for analysing mathematic activity and reconsidering a traditional assumption in mathematic education that there is a great distance between common sense and mathematical logic.

Logical matters are largely underestimated by teacher, as well in secondary school as at tertiary level, in particular concerning a great difference between novices and experts:

An expert in a mathematical field knows when it is dangerous to slack off the rigorous application of inference rules, while novices have to learn this at the same times as they acquire the relevant mathematical knowledge. These two aspects of mathematics cannot be learned separately.

The back and forth between sentences and objects, syntax and semantics, interpretation and theory is at the core of mathematical learning.

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Thank you for your attention

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