Liquid Tensor Experiment

A case study in epistemology of proof

Joint work with

- Peter Scholze

The Lean community

- Adam Topaz
- Riccardo Brasca
- Patrick Massot
- Scott Morrison
- Kevin Buzzard
- Bhavik Mehta
- Filippo A.E. Nuccio
- Andrew Yang
- Damiano Testa
- Heather Macbeth
- Mario Carneiro
- many others

Peter Scholze (5 Dec 2020)

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Check the main theorem of liquid vector spaces

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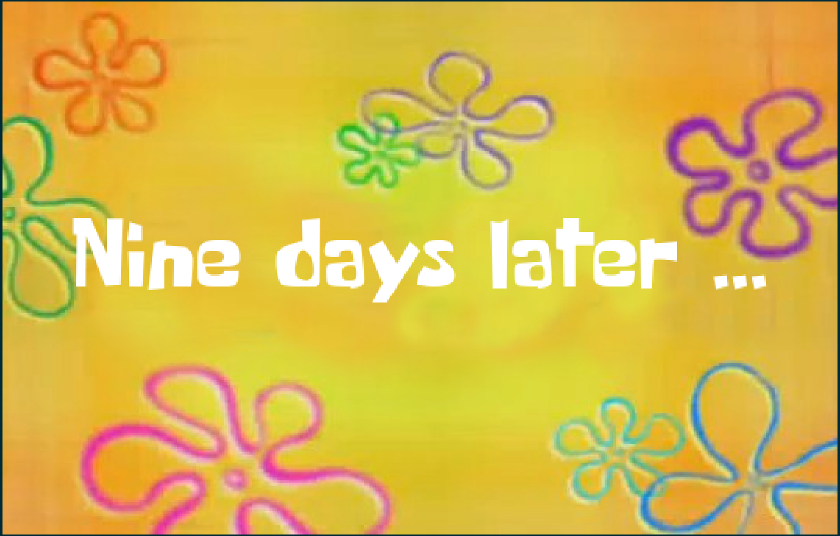
Check the main theorem of liquid vector spaces

…on a computer



Credit: https://spongebob.gavinr.com/

Credit: https://twitter.com/Jcrudess/status/1338922029278441483/photo/1

1998 Liquid Tension Experiment

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1999 Liquid Tension Experiment 2

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2020 Dec 05: “Liquid Tensor Experiment”, Peter Scholze

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tracks include: *Solid Resolution Theory*, *Beating the Odds*, *Shades of Hope*

Theorem (Clausen–Scholze)

Let $0 < p' < p < 1$ be real numbers,
let S be a profinite set,
and let V be a p-Banach space.

Let $\mathcal{M}_{p'}(S)$ be the space of p'-measures on S.

Then

$$\operatorname{Ext}^i_{\mathrm{Cond(Ab)}}(\mathcal{M}_{p'}(S), V) = 0$$

for $i \geqslant 1$.

Why did Scholze want a formalization?

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10 reasons. (Emphasis by jmc, unless indicated.)

Scholze, LTE blogpost 2020, reason 1

"I want to make the strong claim that in the foundations of mathematics, one should replace topological spaces with condensed sets (except when they are meant to be topoi — topoi form a separate variant of topological spaces that is useful, and somewhat incomparable to condensed sets). This claim is only tenable if condensed sets can also serve their purpose within real functional analysis."

Scholze, LTE blogpost 2020, reason 2

"[W]ith this theorem, the hope that the condensed formalism can be fruitfully applied to real functional analysis stands or falls. I think the theorem is of utmost foundational importance, *so being 99.9% sure is not enough*."

Scholze, LTE blogpost 2020, reason 3

"[I]f it stands, the theorem gives a powerful framework for real functional analysis, making it into an essentially algebraic theory. For example, in the Masterclass, Clausen sketched how to prove basic results on compact Riemann surfaces or general compact complex manifolds (finiteness of cohomology, Serre duality), and one can black box all the functional analysis into this theorem. Generally, whenever one is trying to mix real functional analysis with the formalism of derived categories, *this would be a powerful black box. As it will be used as a black box, a mistake in this proof could remain uncaught.*"

Scholze, LTE blogpost 2020, reason 4

“I spent much of 2019 obsessed with the proof of this theorem, almost getting crazy over it. In the end, we were able to get an argument pinned down on paper, *but I think nobody else has dared to look at the details of this, and so I still have some small lingering doubts.”*

Scholze, LTE blogpost 2020, reason 5

"[A]s I explain below, the proof of the theorem has some very unexpected features. In particular, it is very much of *arithmetic* [emphasis by Scholze] nature. *It is the kind of argument that needs to be closely inspected.*"

Scholze, LTE blogpost 2020, reason 6

“[W]hile I was very happy to see many study groups on condensed mathematics throughout the world, *to my knowledge all of them have stopped short of this proof.* (Yes, this proof is not much fun…)”

Scholze, LTE blogpost 2020, reason 7

"I have occasionally been able to be very persuasive even with wrong arguments. (Fun fact: In the selection exams for the international math olympiad, twice I got full points for a wrong solution. Later, I once had a full proof of the weight-monodromy conjecture that passed the judgment of some top mathematicians, but then it turned out to contain a fatal mistake.)*"*

Scholze, LTE blogpost 2020, reason 8

"[T]he Lean community has already showed some interest in formalizing parts of condensed mathematics, so the theorem seems like a good goalpost."

Scholze, LTE blogpost 2020, reason 9

"[F]rom what I hear, it sounds like the goal is not completely out of reach. (Besides some general topos theory and homological algebra (and, for one point, a bit of stable homotopy theory(!)), the argument mostly uses undergraduate mathematics.) If achieved, it would be a strong signal that a computer verification of current research in very abstract mathematics has become possible. I'll certainly be excited to watch any progress."

Scholze, LTE blogpost 2020, reason 10

"I think this may be my most important theorem to date. (It does not really have any applications so far, but I'm sure this will change.) *Better be sure it's correct…*"

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- After 1 month, I had formalized the statement of Theorem 9.4. $\implies$ Start a public project.
- $\approx$ 15 people contributed. Many online discussions with Scholze.
- 6 months after Scholze’s blogpost, Theorem 9.4 was formalized.

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- Alternative to Breen–Deligne resolutions

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He reflects on several questions about the project.

Scholze's 2nd blogpost, excerpt 1

Question: What is the significance of Theorem 9.4? Is the Liquid Tensor Experiment completed?

Answer: Theorem 9.4 is an extremely technical statement, whose proof is however the heart of the challenge, and is the only result I was worried about. So with its formal verification, I have no remaining doubts about the correctness of the main proof. Thus, to me the experiment is already successful; but the challenge of my blog post has not been completed. [...]

Scholze's 2nd blogpost, excerpt 2

Question: Was the proof in [Analytic.pdf] found to be correct?

Answer: Yes, up to some usual slight imprecisions.

Scholze's 2nd blogpost, excerpt 3

Question: Were any of these imprecisions severe enough to get you worried about the veracity of the argument?

Answer: One day I was sweating a little bit. [. . .] There was one subtlety related to quotient norms [. . .] that was causing some unexpected headaches. But the issues were quickly resolved, and required only very minor changes to the argument. Still, this was precisely the kind of oversight I was worried about when I asked for the formal verification.

Scholze's 2nd blogpost, excerpt 4

Question: Were there any other issues?

Answer: There was another issue with the third hypothesis in Lemma 9.6 (and some imprecision around Proposition 8.17); it could quickly be corrected, but again was the kind of thing I was worried about. The proof walks a fine line, so if some argument needs constants that are quite a bit different from what I claimed, it might have collapsed.

Scholze's 2nd blogpost, excerpt 5

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Comment by jmc: My understanding was strongly aided by Lean, and grew gradually throughout the project.

Scholze's 2nd blogpost, excerpt 6

Question: Interesting! What else did you learn?

Answer: What actually makes the proof work! When I wrote the blog post half a year ago, I did not understand why the argument worked, and why we had to move from the reals to a certain ring of arithmetic Laurent series. But during the formalization, a significant amount of convex geometry had to be formalized (in order to prove a well-known lemma known as Gordan's lemma), and this made me realize that actually the key thing happening is a reduction from a non-convex problem over the reals to a convex problem over the integers. [. . .]

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The story continues ...

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- This took us another year.

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Some intermediate statements rely on

- zero-width unicode chars:
  to transparently trigger some automation
  while keeping the statement readable

An alignment problem!

How do we know that this formal proof is an analogue of the informal proof?

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Manually checking all the definitions and notations is not realistic.

Abduc(k)tive reasoning

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If it looks like a duck,
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then it probably is a duck.

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- Companion blogpost:
  `https://leanprover-community.github.io/blog/posts/lte-examples/`

Scholze, after completion of the project (1)

"I guess my perspective here was that for this project, it's unfeasible to check the definitions. To a human it's easier to convince themselves that there are enough ideas in there to get a proof, than to say with absolute certainty that all the definitions entering the theorem statement are correct.

[. . .]

Scholze, after completion of the project (2)

> What happened in the first half of LTE is that I could really see how you were following the manuscript line-by-line and, in the process of carefully translating it into Lean, catching a number of small slips. This type of process is certainly something where formal proof verification is doing an excellent job, *and it really radically changed my confidence in the argument*. However, the second half of LTE has not added anything to my confidence (but it was already at 100% ;-) , so maybe that's not saying much). But of course it's a really impressive achievement to get all this mathematical machinery done in Lean!"