TECHNOLOGY-SUPPORTED LEARNING OF PROOFS IN MATHEMATICS

Introduction to a didactic approach of Math PATeaching

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Sources for the preparation of the course are given at the foot of the transition slides, references to specific points are given on the relevant slide with a bibliography after the final slide. Most of the relevant references for the theme of this course can be found in the source documents.

→ Historical milestones in the teaching of proof

Source
Balacheff N. (to be published) Notes for a study of the didactic transposition of mathematical proof. *Philosophy of Mathematics Education Journal*. University of Exeter, United Kingdom.

(1) History in a nutshell, the origin

Euclid, seed of proof in mathematics

- The seed of modern proof in maths: Euclid (circa 300 BC)
  - the ideal of rigour of a theoretical foundation
  - Inseparable from Geometry: *duality of reasoning and perception*
- The Euclidean norms in force until the middle of the 20th century
- Critics:
  - Descartes (1628): the Euclidean ideal favours conviction over understanding.
  - *Dechalles* (1709): to add indications of the use of each proposition!
  - *Clairaut* (1741): to make use of practical geometry to provide meaning, but refuses to be a treatise on surveying. Clairaut refuses to demonstrate the obvious.

Until the French Revolution, Euclid's geometry was taught to a privileged class of the population, those "who dabbled in mathematics";

After the Revolution, this teaching became part of a national education project that took shape in the first half of the 19th century.

(2) History in a nutshell, the 19th century)

Birth of "public" education

The massive expansion of education gave rise to the need for basic textbooks

→ first didactic transpositions (ref. to institutions)

Break between practical and theoretical geometry (public education project, 1793)

Geometry close to Euclid's model is taught to those (men) who want to go on to university.

- Legendre (1794): « efforts to give the '*démonstrations'* all the clarity and brevity that the subject requires » → a said « rival of Euclid »
- Lacroix (1798) to promote understanding in relation to applications, to link understanding and proving, national reference, his book is a *textbook*.

Legendre and Lacroix define the meta-terms: *theorem* « which become obvious through *demonstrations »* (Legendre), « statements which must be *demonstrated »*(Lacroix).

→ but *démonstration itself* is not one of the defined terms.

At the end of the 19th century, proof was a named object: *the "démonstration" **

It was not an object of teaching.

(*) The English word Demonstration was replaced in curricula by the word Proof in the course of the 20th century

(3) History in a nutshell, 1st half of the 20th century

Mathematician, a profession

Economic and industrial development

Birth of the mathematics community

- 1893 separation from astronomy
- 1897 first congress of mathematicians
- 1899 creation of "L'enseignement mathématique".
- 1908 creation of ICMI
- 1911 a report on degrees of rigour
  - A) *Entirely logical method ;* B) *Empirical foundations, logical development ;* C) *Intuitive considerations alternate with the deductive method* ; D) *Intuitive-experimental* method
- 1920 Creation of the IMU in Strasbourg
- 1969 First ICME conference in Lyon
- 1977 First PME conference in Utrecht

Proof has the status of a mathematical tool to be taught

It is taught through geometry.

(4) History in a nutshell, 2nd half of the 20th century

Proof emancipates itself from geometry

Mathematics is

- present in all areas of science (natural sciences, humanities social sciences)
- seen as a universal language

Mathematics from Kindergarten to University (e.g. APMEP 1967)*

→ Period 1, modern mathematics :

- Mathematics is a deductive science, not an experimental one.
- Mathematics is a theory that must bring together under a single structure knowledge that was previously presented in a scattered manner.

→ Period 2, giving up on modern mathematics (70s in the USA, early 80s)

→ emphasis on problem solving, applications of the discipline.

No return to Euclid

Since the early 2000s: **make mathematics the place for genuine research:** *develop the ability to reason and argue, experiment and imagine.*

Mathematical proof is losing ground to deductive reasoning,

→ a broader concept of validation in mathematics teaching

(*) French association of mathematics teachers

(5) History in a nutshell, the 21st century

Proof from kinderkarten to university

TIMSS distinction between...

- **content domains:** specific mathematics subjects matter
- the cognitive domains: sets of expected students' behaviors

Reasoning → *making deductions based on specific assumptions and rules, and justifying results*.

TIMSS Assessment framework documents, grades 4 and 8 (càd CM1 & 4°)

2003	Justify/Prove	"Provide proof for the validity of an action or the truth of a statement by reference to mathematical results or properties; develop mathematical arguments to prove or disprove statements, given relevant information." (TIMSS 2003 p. 33)
2007	Justify	"Provide a justification for the truth or falsity of a statement by reference to mathematical results or properties" (TIMSS 2007 p. 38).
2019	Justify	"Provide mathematical arguments to support a strategy or solution." (TIMSS 2019 p. 24)

→ A glimpse of the Technology Enhanced Learning (TEL) of proof history

Source
Anderson, J. R., Corbett, A. T., Koedinger, K. R., & Pelletier, R. (1995). Cognitive tutors: Lessons learned. The journal of the learning sciences, 4(2), 167-207.
Balacheff N., Boy de la Tour T. (2019) Proof Technology and Learning in Mathematics: Common Issues and Perspectives. In: Hana G., Reid D., de Villiers M. (eds) Proof Technology in Mathematics Research and Teaching (pp. 349-365). Berlin: Springer.

(1) ATP & precursors of proof tutors (the 50's)

Focus on modeling human reasoning with a view to implementing mathematical logic.

- The Logic Theorist based on the Principia Mathematica, on mathematicians' introspection observation of students solving proof problems in symbolic logic
- The General Problem Solver, an improved model to face demanding challenges:
  "problem solving is the battle of selection techniques against a space of possibilities that keeps expanding exponentially" (Simon, Shaw, Newell).
- The Geometry Machine, limited domain and search for specifics heuristics that could model the discovery of proofs in elementary Euclidean plane geometry with a pragmatic approach:
  [T]he machine is granted the same privileges enjoyed by the high-school student who is always assuming (i.e., introducing as additional axioms) the truth of a plethora of 'obviously self-evident' statements concerning, for example, the ordering properties of points on a line and the intersection properties of lines in a plane. (Gelernter)

Several issues:

- the depth of the gap between machine and human ways of reasoning,
- the multiplicity of representations in geometry (reasoning based on visualization)
- the constraints on human–machine communication

The convergence of ATP and educational technology failed short.

(2) Proof tutors, the cognitive approach (the 70-90's)

Building a theorem prover is an exciting alternative to the usual classroom presentation (Goldstein)

- Representations of procedural knowledge
- Objective to implement a 'natural' formalism of mathematical knowledge.
- Reasoning strategy: model backward Vs forward model

prepared the ground for Geometry Tutor (Anderson & al.1985)

- based on theoretical analysis of the characteristics of the problem domain and on empirical observation of students' behavior
- It included
  - "ideal models" in order to generate proofs in a natural way
  - "buggy models" to diagnose student errors as production rules
  - immediate feedback to keep students on the right track

Cognitive tutors include (Anderson & al.1995).

- cognitive model of the content to be learned
- scaffold strategies for feedback and advice The Angle Project (Koedinger & al. 1990) which gave a more active role to diagrams

(3) Proof tutors, the emergence of the interface

ATP research focused on computational models

the Geometry Tutor project recognizes that

- "**placing interface design ahead of production–systems design represents a major restructuring of our approach to tutor construction**" (Anderson et al., 1995, p.35).
- Beneath the interface, the implemented models must ensure that communication makes representations accessible to students and enables interactions that enhance learning

Long-term goal to build an intelligent tutoring system for elementary geometry:

Any proofs and constructions found by our automated geometry theorem prover [GRAMY] must be stated with the common ontology of the axiomatized geometry system taught in schools. (Matsuda & VanLehn)

- To comply with the constraints of the interface, and the professional responsibilities of teachers.
- The desired geometry theorem prover must not only be able to find a single comprehensible proof, it should also be able to find all proofs that are considered acceptable to instructors

The concept of microworld

1960s: Minsky began thinking about modelling knowledge and reasoning.

He invited Papert to join him at MIT in 1963.

AI research combines computer and cognitive modelling

1970s: the concept of microworld is born of the idea that:

- Rather than looking for a general representation, develop "mini-theories" that could be articulated to model complex phenomena
- New math occults The mathematical experience
- Responsibility of computer scientists :

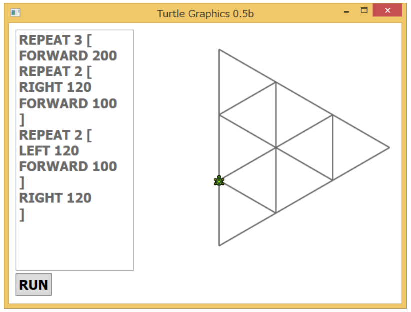
> The computer scientist is the proprietor of the concept of procedure, the secret educators have so long been seeking (Minsky, 1969).

The microworld concept is the product of

- AI research (vision and robotics, the concept of mini-theory)
- A critical reflection on the New Math movement
- A principle: modelling should focus on action and not on logical formalization.

LOGO is the first proof of concept

Microworlds and Proof



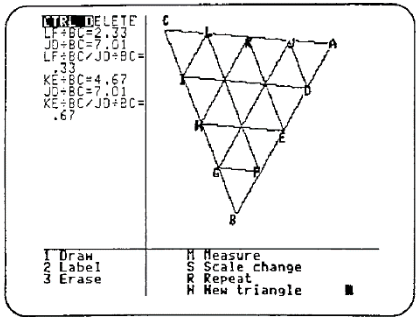
Microworlds provide students with a field of experience for…

- Exploring mathematical facts
- Making conjectures
- Shaping argumentations
- Proposing proofs

(Boero et al. 1995, Baccaglini-Frank & Mariotti, 2010).

Logo

Programming language with visual feedback
Language primitives: move, turn
Differential geometry

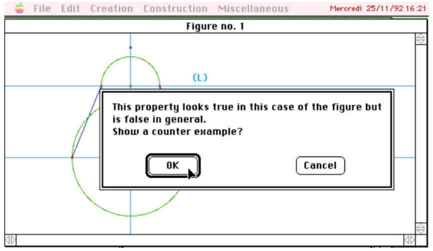


Geometric supposer,

"An intellectual prothesis for making conjectures"
Rule and compass, Euclidean geometry
Help understand that a picture is a special case, part of a larger process

Cabri-géomètre

Construction with tools implementing Euclidean geometry objects and properties (instrumental axioms)
Maintains properties in direct manipulation
Visual feedback (invariance) and oracles



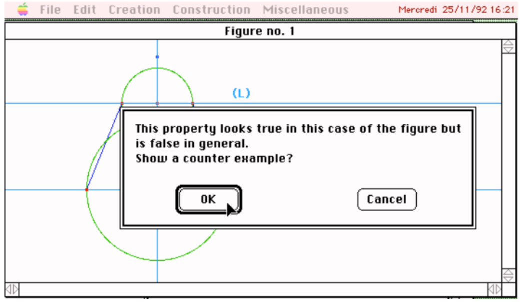
Dynamic geometry environments
DGE

ATP augmented microworlds

DGE are not illustrations but “living” objects within an evolving world

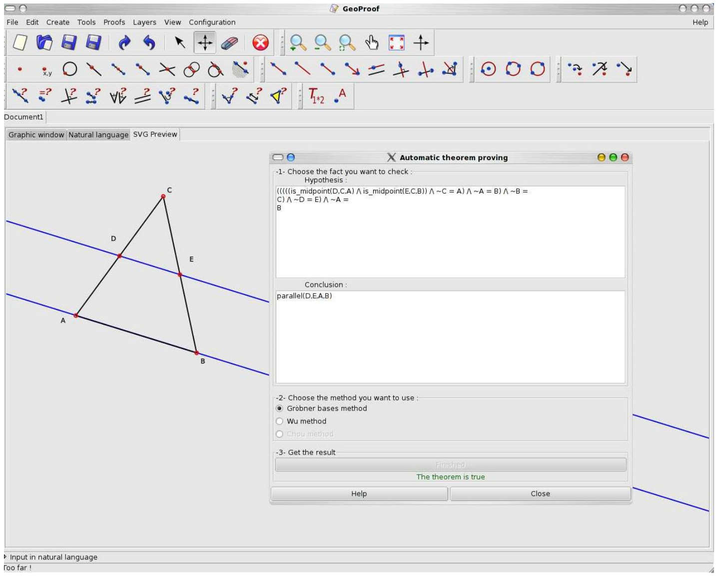
ATP can provide services for…

- Providing counter examples or textual critics
- Providing feedback on a textual representation of a solution
- feedback of a textual or a visual form
- scaffolding the writing of a proof,



e.g. GeoGebra “[can answer] a query posed by a user about the truth or falsity of any geometric statement” or “present further hypotheses that should be considered for the proposition to become true" (Hauer, Kovács, Recio, & Vélez, 2018, p.2).

- Cabri (Laborde 1990)
- Cabri-Euclide (Luengo, 1997)
- Baghera (Balacheff et al. 2000)
- Agent-geom (Richard 2007)
- Geogebra (Hohenwarter 2002)
- GeoProof (Narboux 2006)
- …/…



Lesson 1: formal versus informal mathematics

The history of the teaching of proof and of the contribution of educational technology suggests

- An initial “common sense” idea that teaching the norm will improve understanding the role of proof as a concept and a tool
- An evolution which search for linking the norm of the discipline and the meaning of its object, mathematics

A controversy about the relationships between formal and informal proof:

- (1) the claim that “There is a formal analogue of a purported informal mathematical proof or else the latter fails to be a proof.” (Azzouni, 2009, p. 14),
- (2) the claim that “mathematical proofs are cemented via arguments based on the meaning of the mathematical terms that occur in them, which by their very conceptual nature cannot be captured by formal calculi.” (Rav, 2007, p. 294).

« Every mathematician knows that a proof is not truly "understood" as long as one has limited oneself to verifying step by step the correctness of the deductions that appear in it, without trying to clearly conceive the ideas that led to the construction of this chain of deductions in preference to any other. » (Bourbaki, 1948, p. 37 n.1)

Lesson 1: deduction and meaning

Formal in mathematics, distinguishes three facets:

- Facet 1 “concerns the form of mathematical sentences, as structured syntactical objects independently of their intertextual contexts”
- Facet 2 “concerns the way mathematics is presented as a final product, in a formalised language, generally contrasted with that after which results are found by mathematicians”
- Facet 3 “concerns the very notion of logical consequence”

(Arzarello 2007 p,43)

The essence of the issue is the relationships between

- conceptual proofs with a semantic content, usual in practice

and

- derivations, that is syntactic objects of some formal system

→ Lexical and conceptual clarification

Source:
Balacheff N. (2021) The transition from mathematical argumentation to mathematical proof, a learning and teaching challenge. The 14th International Congress on Mathematical Education. Shanghai, 12th –19th July, 2020

Choosing the words

Reasoning

Organization of statements that is oriented towards modifying the epistemic value of a target statement

→ The need to construct of coherence and belonging of the new statement to the knowledge system (the known)

Explanation

A system of relationships within which the target statement finds its place with respect to the known. It establishes the validity of the target statement

Argumentation is a discourse

Oriented - it aims at the validity of a statement

Intentional - it seeks to modify a judgment

Critical - it analyzes, it supports or defends

Proof

An argumentation accepted by a given community at a given time.

It requires a system of validation common to the interlocutors.

Mathematical proof (fr. *démonstration*)

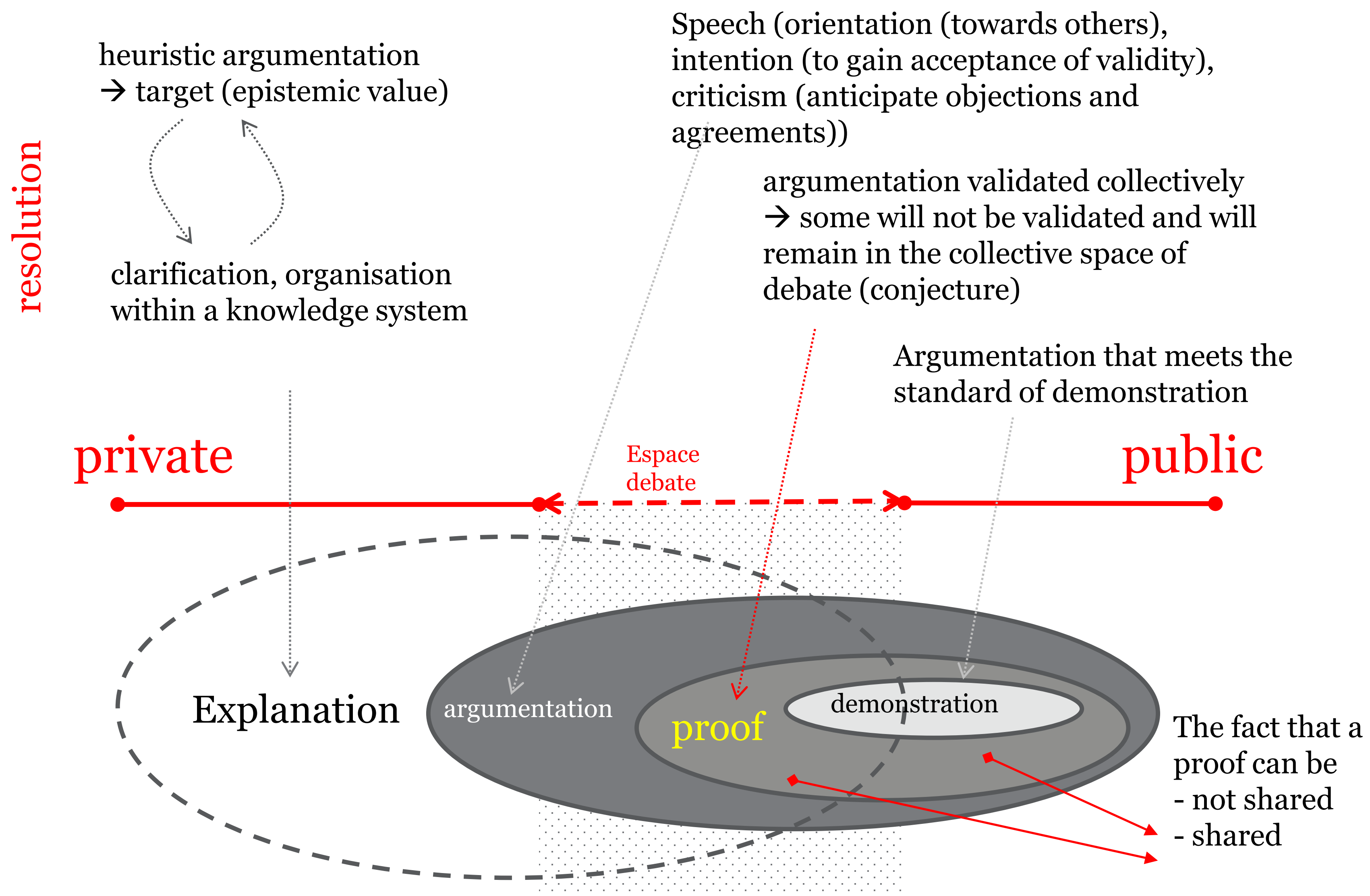
An argumentation structured according to rules conforming to a standard established by mathematicians

a statement is known as true, or is being true, or is deduced from those which precede it with the help of an inference rule taken from a well-defined set of rules.

Formal proof

"A **formal proof** is a finite sequence of well-formed expressions following a strict grammar and a vocabulary devoid of ambiguity, each of which is an axiom, an assumption, or follows from the preceding sentences in the sequence by a rule of inference."

Explanation – Argumentation – Proof

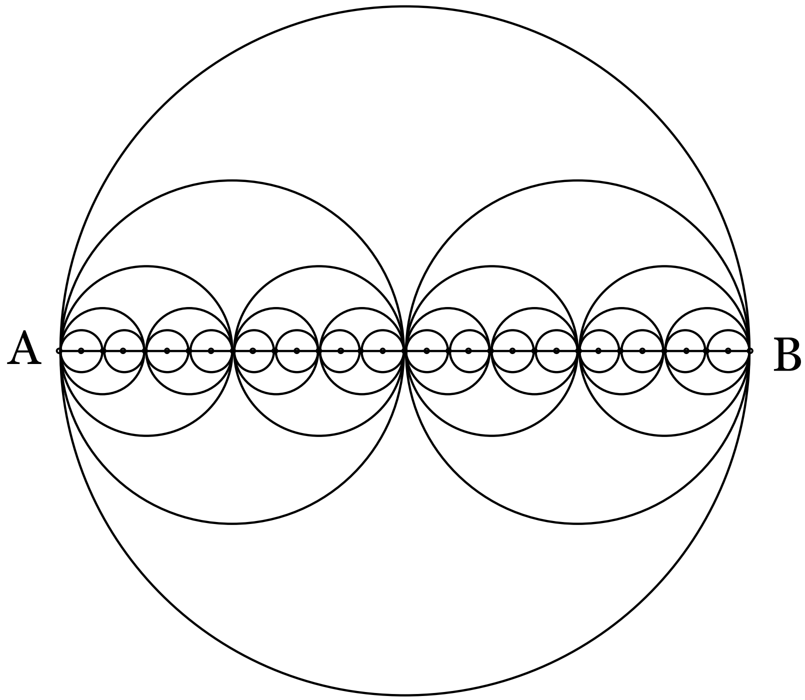


→ Interlude… two cases to ponder

Source:
Balacheff N. (2017) cK¢, a model to understand learners' understanding – Discussing the case of functions. In: El calculo y su ensenanza, 2017, IX (Jul-Dic), pp.1-23
Pedemonte, B., & Balacheff, N. (2016). Establishing links between conceptions, argumentation andproof through the ck¢-enriched Toulmin model. The Journal of Mathematical Behavior, 104-122.

Example 1, the case

31. Vincent : the area is always divided by 2...so, at the limit? The limit is a line, the segment from which we started ...
32. Ludovic : but the area is divided by two each time
33. Vincent : yes, and then it is 0
34. Ludovic : yes this is true if we go on...
37. Vincent : yes, but then the perimeter ... ?
38. Ludovic: no, the perimeter is always the same
41. Vincent: It falls on the segment… the circles are so small.
42. Ludovic: Hmm… but it is always 2πr.
43. Vincent: Yes, but when the area tends to 0 it will be almost equal…
44. Ludovic: No, I don’t think so.
45. Vincent: If the area tends to 0, then the perimeter also… I don’t know…
46. Ludovic: I will finish writing the proof.



Construct a circle with AB as a diameter. Split AB in two equal parts, AC and CB. Then construct the two circles of diameter AC and CB... and so on.

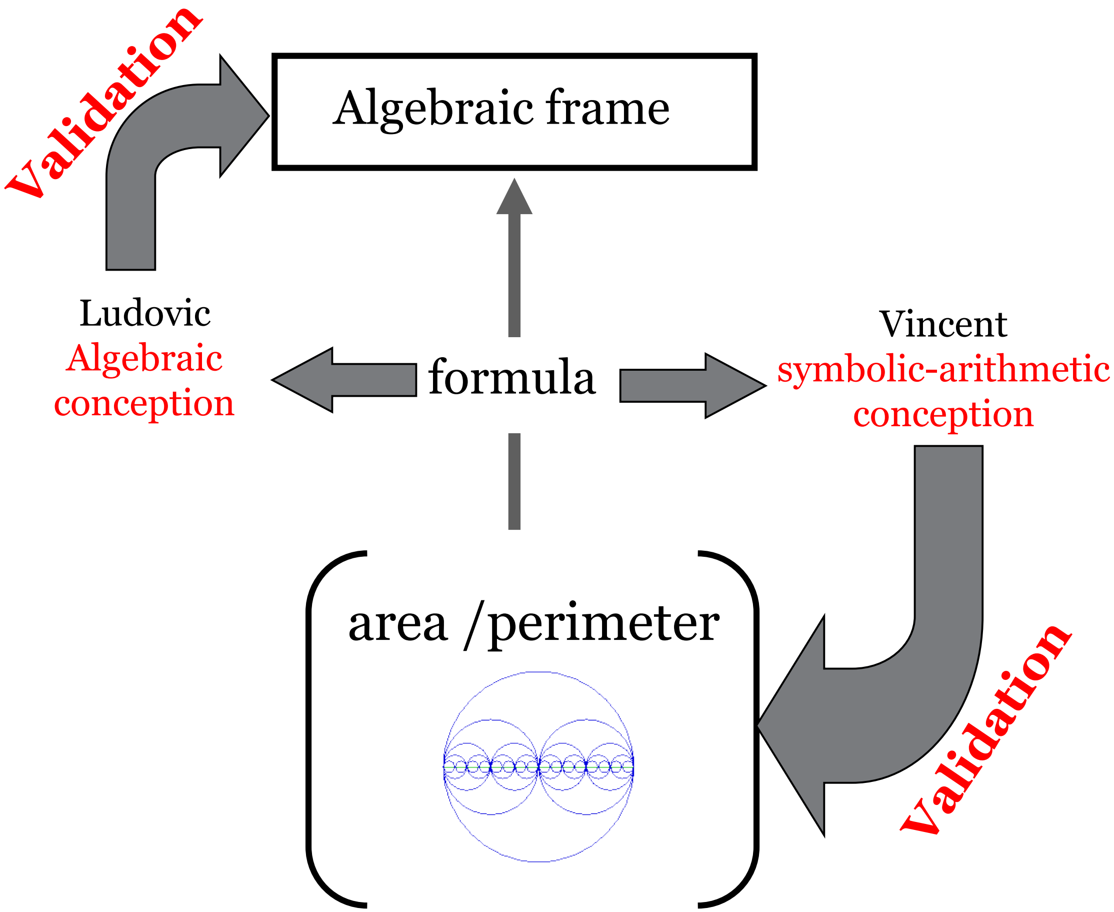
How does the perimeter vary at each stage?

How does the area vary?”

Example 1, analysis

The symbolic representation works as a *boundary object* adapting the different meanings but being robust enough to work as a tool for both students.

The differences lie in the control grounding their activity.



Case 2: series of continuous functions

(1) $u_0, u_1, u_2 \ldots u_n, u_{n+1}$, &c. . . .

Lorsque, les termes de la série (1) renfermant une même variable x, cette série est convergente, et ses différens termes fonctions continues de x, dans le voisinage d'une valeur particulière attribuée à cette variable ;

$$s_n, \quad r_n \quad \text{et} \quad s$$

sont encore trois fonctions de la variable x, dont la première est évidemment continue par rapport à x dans le voisinage de la valeur particulière dont il s'agit. Cela posé, considérons les accroissemens que reçoivent ces trois fonctions, lorsqu'on fait croître x d'une quantité infiniment petite α. L'accroisse-

x is not explicit in the writing

Lorsque des quantités variables sont tellement liées entre elles que, la valeur de l'une d'elles étant donnée, on puisse en conclure les valeurs de toutes les autres, on conçoit d'ordinaire ces diverses quantités exprimées au moyen de l'une d'entre elles, qui prend alors le nom de *variable indépendante ;* et les autres quantités exprimées au moyen de la variable indépendante sont ce qu'on appelle des fonctions de cette variable.

u_n and x are two variables, but x is the independent variable on which depends u_n

→ the notion of function can be both practically close to the modern one and conceptually reflect the dominant understanding of the time

Case 2: series of continuous functions

x d'une quantité infiniment petite α. L'accroissement de s_n sera, pour toutes les valeurs possibles de n, une quantité infiniment petite; et celui de r_n deviendra insensible en même temps que r_n, si l'on attribue à n une valeur très-considérable. Par suite, l'accroissement de la fonction s ne pourra être qu'une quantité infiniment petite. De cette remarque on déduit immédiatement la proposition suivante.

1.er Théorème. *Lorsque les différens termes de la série* (1) *sont des fonctions d'une même variable* x, *continues par rapport à cette variable dans le voisinage d'une valeur particulière pour laquelle la série est convergente, la somme* s *de la série est aussi, dans le voisinage de cette valeur particulière, fonction continue de* x.

The notion of limit is controlled by a kind of kinematic "concept image" (inherited from Neper and Newton and common at that time)

→ Cauchy did not pretend that this is a mathematical proof, as he used to do elsewhere in the course, but a *remark*.

The notion of a "infinitely small" is dynamic: an infinitely small variable is a variable which has zero as a limit

Case 2: series of continuous functions

Si l'on nomme n' un nombre entier supérieur à n, le reste r_n ne sera autre chose que la limite vers laquelle convergera, pour des valeurs croissantes de n', la différence

(3) $$s_{n'} - s_n = u_n + u_{n+1} + \ldots + u_{n'-1}.$$

Concevons, maintenant, qu'en attribuant à n une valeur suffisamment grande, on puisse rendre, pour toutes les valeurs de x comprises entre les limites données, le module de l'expression (3) (quel que soit n'), et, par suite, le module de r_n, inférieur à un nombre ε aussi petit que l'on voudra. Comme un accroissement attribué à x pourra encore être supposé assez rapproché de zéro pour que l'accroissement correspondant de s_n offre un module inférieur à un nombre aussi petit que l'on voudra, il est clair qu'il suffira d'attribuer au nombre n une valeur infiniment grande, et à l'accroissement de x une valeur infiniment petite, pour démontrer, entre les limites données, la continuité de la fonction

$$s = s_n + r_n.$$

Mais cette démonstration suppose évidemment que l'expression (3) remplit la condition ci-dessus énoncée, c'est-à-dire que cette expression devient infiniment petite pour une valeur infiniment grande attribuée au nombre entier n. D'ailleurs, si cette condition est remplie, la série (1) sera évidemment convergente. En conséquence, on peut énoncer le théorème suivant :

- the variable *x* remains implicit in the expression [again embedded in the terms of the series]
- the order of the text is not congruent to the logical order it expresses
- *n* depends on *ε* and not on *x*

$$\forall \varepsilon\ \exists N\ \forall x$$

- This is a non-modern expression of the Cauchy criterion of Uniform convergence

$$\forall \varepsilon\ \exists N\ \forall x\ \forall n>N\ \ \forall n'>n\ \ |s_n - s_{n'}| < \varepsilon$$

Case 2: series of continuous functions

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- → A text makes closer to a remark than to a mathematical proof in the modern way.
- → The rigor is there, as a willing, but it encounters obstacles:
  - the algebraic formalism of Calculus is not yet available
  - the kinematic concept image is still dominant in the mathematical community of that time.

but isn't rigor always a willing?

$$\forall \varepsilon\ \exists N\ \forall x\ \forall n>N\ \ \forall\ n'>n\ \ |s_n - s_{n'}| < \varepsilon$$

Example 2: analysis

Gibert Arsac interpretation of Cauchy's understanding is based on a critical and rigorous analysis of the text taking into account the situation of Calculus in the first half of the XIX° century:

1. The notion of variable dominates the notion of function (dependent variable) with a kinematic vision of convergence which impact the concepts of limit and continuity
2. Inequality (<, >) is rarely used and the algebraic notation of absolute value is absent
3. Still under construction, continuity is being defined on an interval and not at a point, tightly linked to a vision of the graphical continuity of a curve.
4. Quantifiers are not in use (one has to wait for the XX° century) making difficult to identify the dependences introduced by their order in a statement, and the negation of a statement which involves them (e.g. discontinuity as a negation of continuity)

From interpretation to modelling

A paraphrase of Henri Poincaré (1905, p,17):

> it is not the mathematical principle of rigour that changes over history, nor the principle of the practice of active mathematicians that differs, but the consequence of the deepening of mathematical knowledge and the evolution of mathematical tools.

Four lines of analysis drive the interpretation and may allow to model the understanding underpinning the case of Cauchy' concept of uniform convergence:

- the nature of the problem addressed (convergence of series of continuous functions)
- the available tools to solve this problem which include those to manipulate rational numbers, variables, function, limit, continuity
- the semiotic systems including natural language, algebraic representation as available at that time, representation of curves
- the controls like the Leibniz law of continuity, the repertoire of known functions

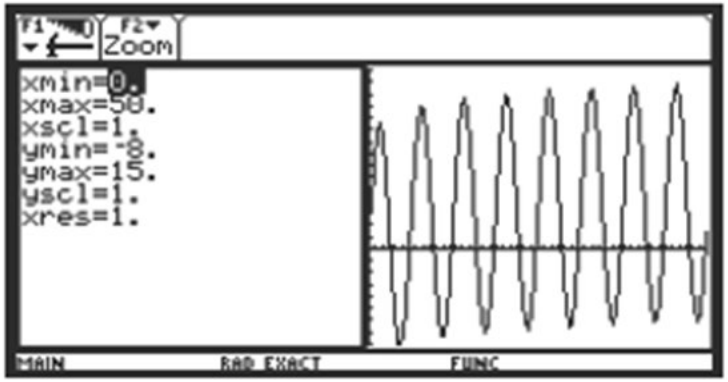
Mathematicians do what they do,
because their objects are what they are

→ Understanding student's understanding

Source:
Balacheff N. (2017) cK¢, a model to understand learners' understanding – Discussing the case of functions. In: El calculo y su ensenanza, 2017, IX (Jul-Dic), pp.1-23
Balacheff N. (2013) cK¢, a model to reason on learners' conceptions [keynote]. In: Proceedings of the PME-NA 2013 Psychology of Mathematics Education, North American Chapter. Chicago, USA

What about misconceptions?

Misconceptions, naïve theories, beliefs have been largely documented in an attempt to make sense of learners' errors and contradictions



« f is defined by $f(x) = \ln(x) + 10\sin(x)$
Is the limit $+\infty$ in $+\infty$? »

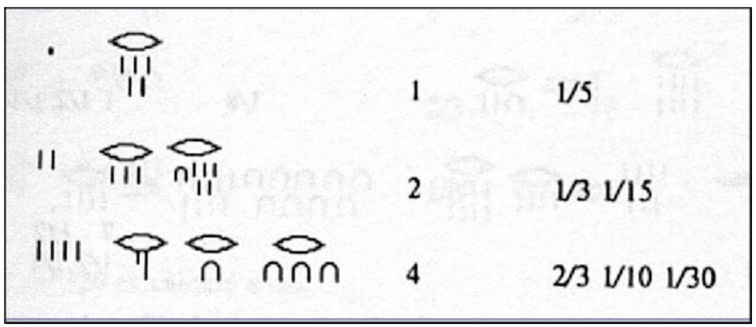
with a graphic calculator 25% of errors
without a graphic calculator 5% of errors
(Guin & Trouche 2001)

Human beings have conceptions which are adapted and efficient in different situations they are familiar with.

- They are not naïve or misconceived, nor mere beliefs.
- They are situated and operational in adequate circumstances.

Interpreting representations

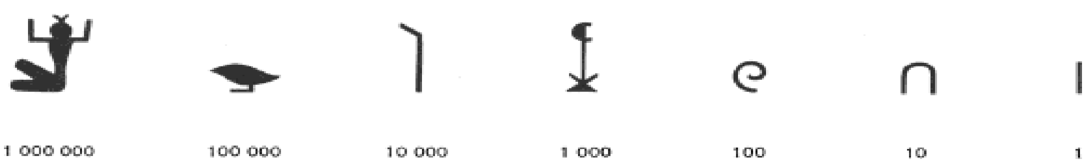
Egyptian computation of 1/5



What is denoted by the signs are parts of the whole, hence integers but integers which could not be added as integers are. Scribes used tables to establish the correspondence between two numbers to be multiplied and to get the result.

For 4055/4093 one will get the shortest and unique additive decomposition:
[1/2 + 1/3 + 1/7 + 1/69 + 1/30650 + 1/10098761225]

Unfortunately, Egyptians could not write the last term.



Addition, from fingers to keystrokes

C1: Verbal counting IIIII & III

P – Quantify union of two sets, objects are physically present, both cardinals are small.

R – match fingers or objects and number names, pointing objects

L – body language, counting

Σ – not counting twice the same, counting all, order of the number names

C 2: Counting on 15+8

P – The numbers are given, but the collections are not present, one of the numbers must be small enough

R – choose the greater number, count-on to determine the result.

L – body language, number naming, verbal counting.

Σ – order of the number names , match fingers to number names

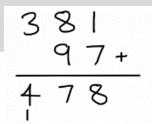
C3: written addition 381+97

P – adding two integers

R – algorithm of column addition

L – decimal representation of numbers

Σ – check the implementation of the algorithm, check the layout of number addition



C4: Pocket calculator

P – adding two integers

R – keystroke to represent a number, to process number addition

L – body language (keystrokes), decimal representation of numbers on the screen

Σ – keystrokes verification, order of magnitude.

These are different conceptions of addition

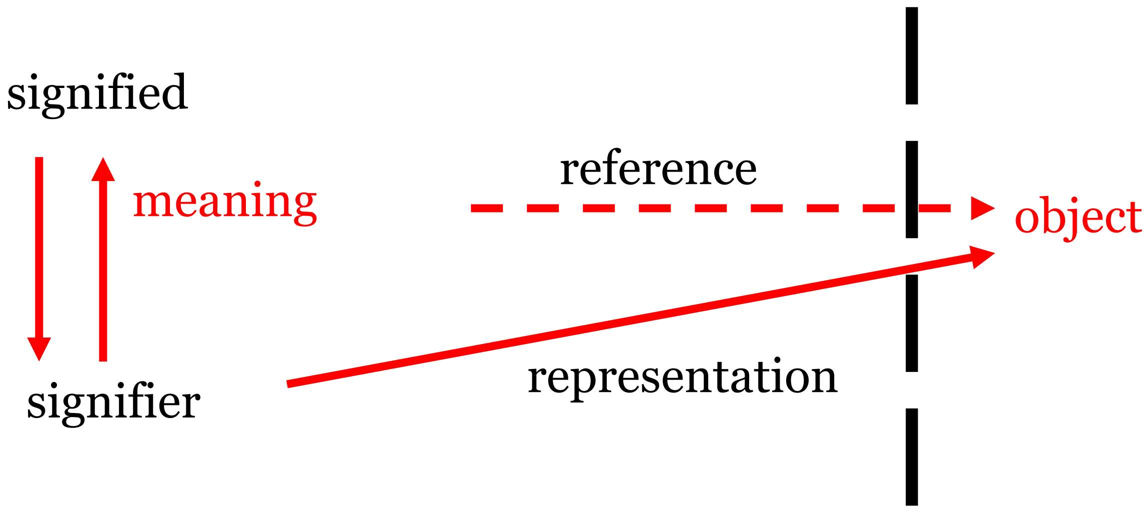
→ Semiotic discussion: object and representation

Source:
Caveing M. (2004) Le problème des objets dans la pensée mathématiques. Paris: Vrin
Duval R. (2018) Understanding the Mathematical Way of Thinking – The Registers of Semiotic Representations. Springer Cham
Vergnaud, G. (1981). Quelques orientations théoriques et méthodologiques des recherches françaises en didactique des mathématiques. Recherche en didactique des mathématiques, 2(2), 215-231.
Tall, D. (2004). Thinking through three worlds of mathematics. Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 4, 281-288.

A mathematical object is never available

A mathematical object is a ‘being’ that is never available through its presence, but through the mediation of a regulated system of designations that make it accessible.

(Maurice Caveing, 2004)



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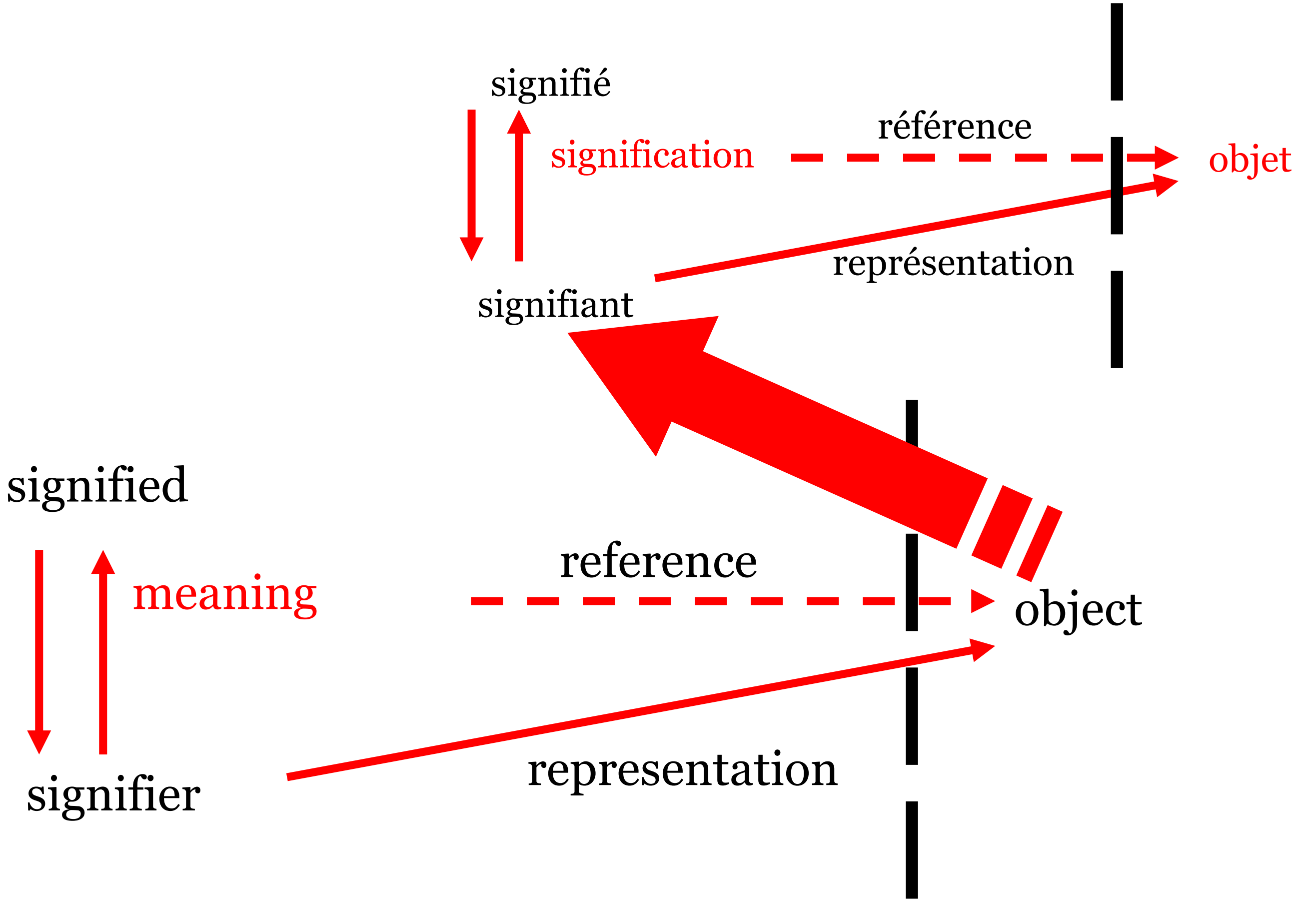
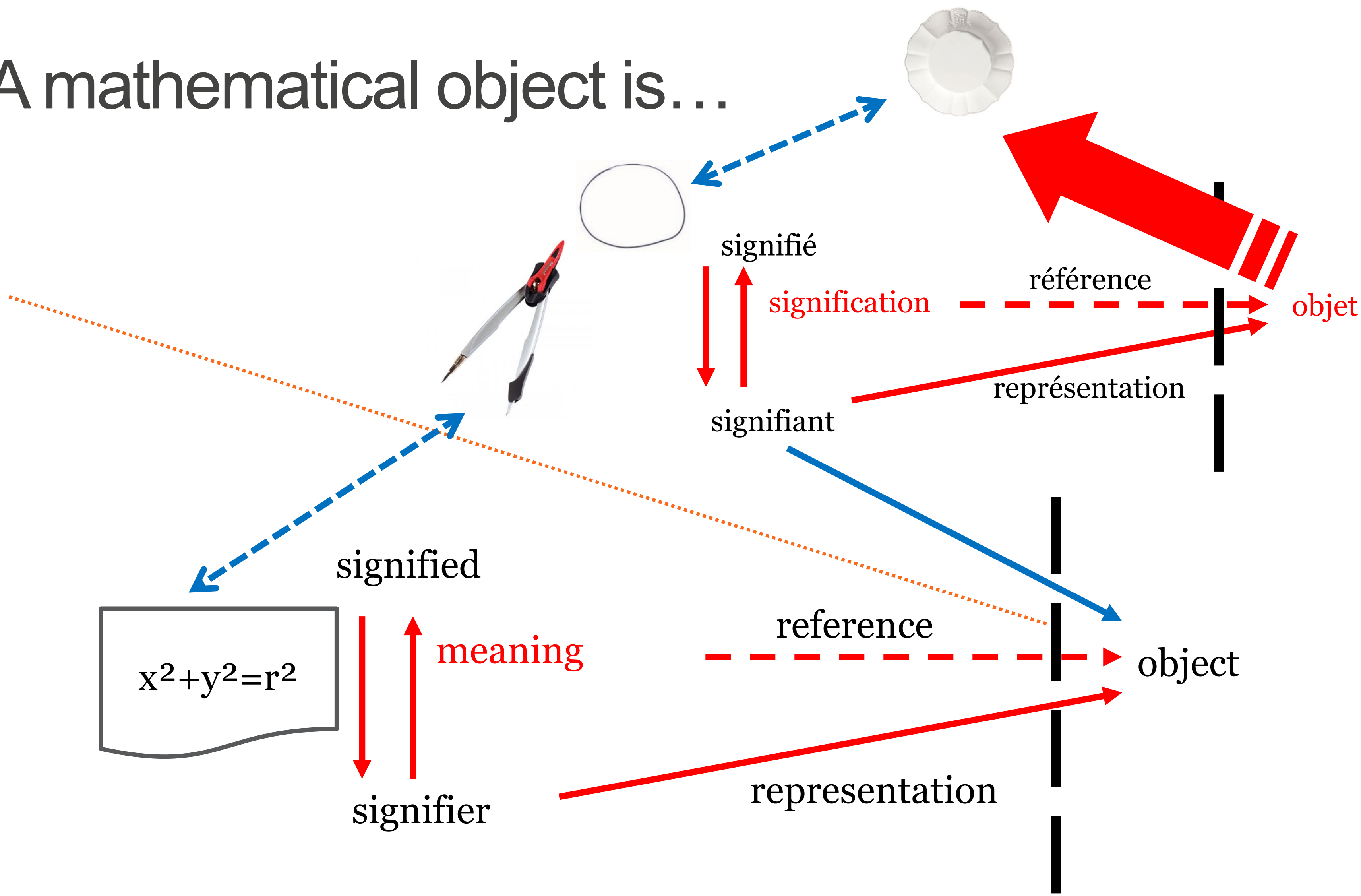


A mathematical object is…

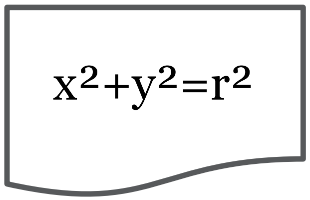


Semiotic register

A mathematical object is a 'being' that is never available through its presence, but through the mediation of a regulated system of designations that make it accessible.
(Maurice Caveing, 2004)

The mathematical object as an invariant between semiotically heterogeneous representations.

Proof assistant



The questioning about the validity of the truth of a statement, takes shape in this semiotic context.

A semiotic register

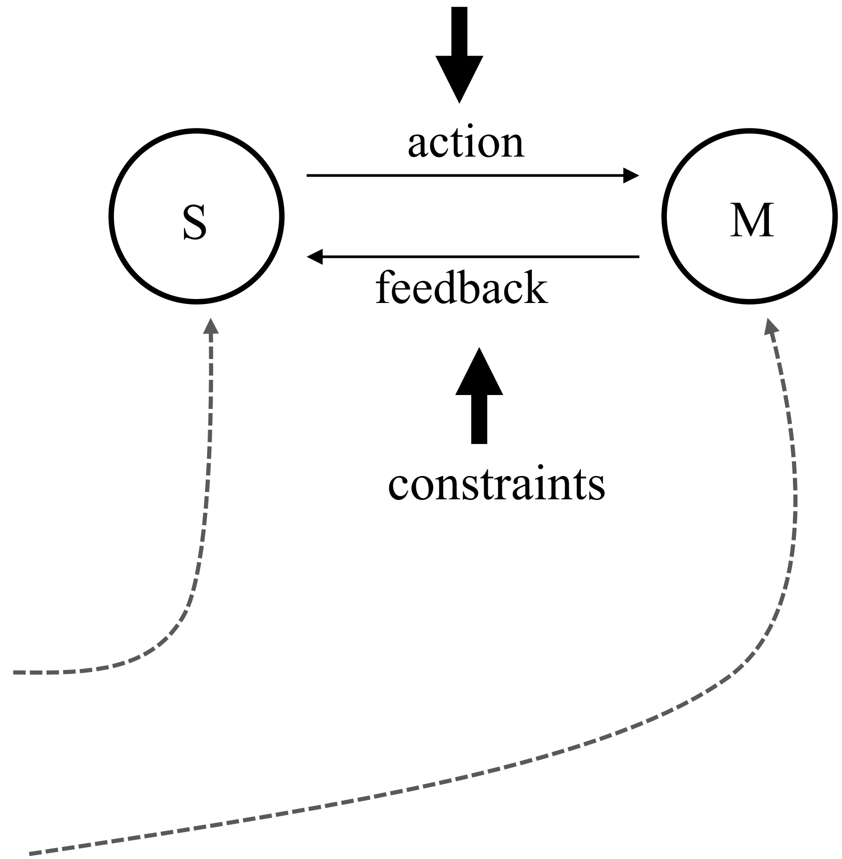
(i) tangible traces identifiable as representations of something,
(ii) transformation rules to produce other representations that can contribute to knowledge,
(iii) conversion rules to another representation system to make other significations explicit,
(iv) conformity rules for constituting higher-level units that can contribute to the evolution of knowledge

Semiotic registers are a key component of a conception of a mathematical object

Modeling mathematical conceptions

- Learner are first persons with their emotions, social commitments, imagination, personal history, cognitive characteristics. They live in a complex environment which has physical, social and symbolic characteristics.

- For the sake of the modeling objective and with in mind the practical limitations it will entails…
- Learners are considered here as the
  - epistemic subjects
- The environment is reduced to those features that are relevant from an epistemic perspective:
  - the milieu
- *the learner's antagonist system in the problem solving and learning process*



(Balacheff 1995, 2013)

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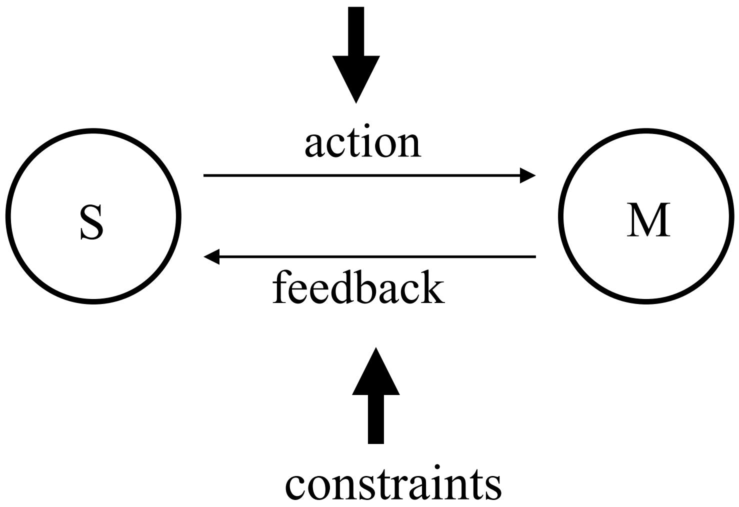
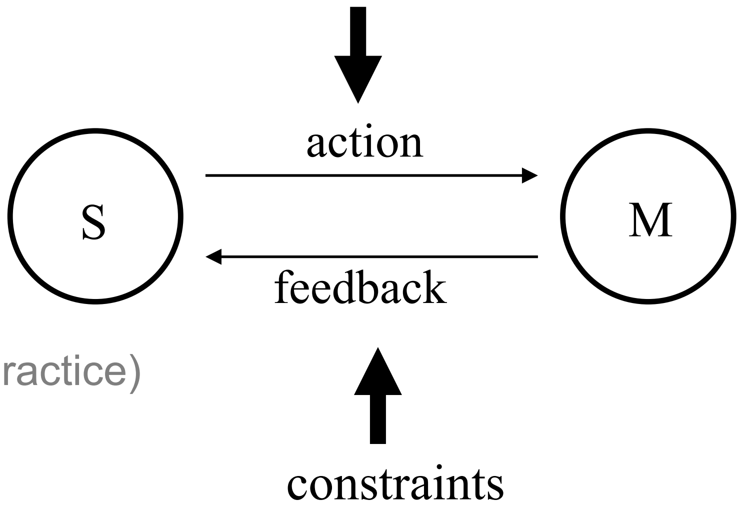


A conception is the state of dynamical equilibrium of an action/feedback loop between a learner and a milieu under proscriptive constraints of viability

(Balacheff 1995, 2013)

Modeling mathematical conceptions

a *conception* is characterized by a quadruplet (P, R, L, Σ) where:



- P is a set of problems (sphere of practice)
- R is a set of operators.
- L is a representation system

  (which may be a semiotic register)
- Σ is a control structure.

the quadruplet is not more related to S than M: the representation system allows the formulation and use of operators by the active sender (the user) as well as the reactive receiver (the milieu); the control structure allows assessing action (S), as well as selecting a feedback (M).

(Balacheff 1995, 2013)

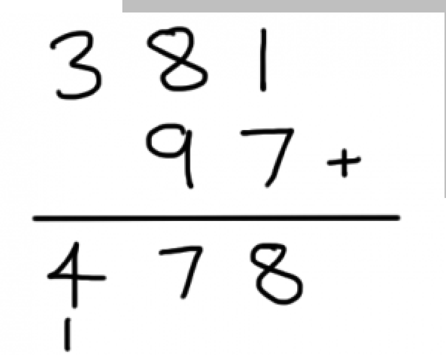
Addition, from fingers to keystrokes

set of problems → problems for which the conception provides efficient means

operators → actions at the interface of the learner/milieu system;

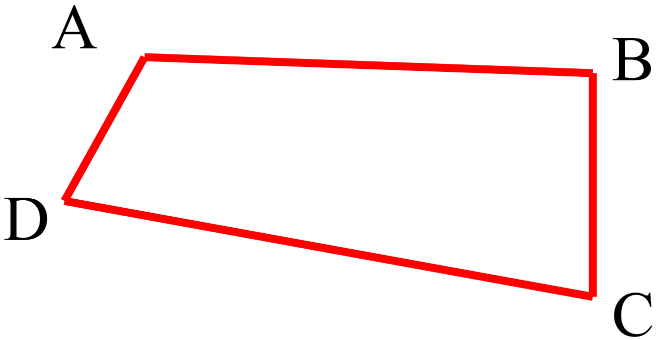
representation system → semiotic means to represent problems, support interaction and represent operators

control structure → making choices, assessing action and feedback, taking decisions, judging the advancement of the problem or task



These are different conceptions of addition

Controls and representation



S = [(AB+DC)/2]x[(AD+BC)/2]

A method used by sugar-cane farmers in Brazil to find the areas of their fields were to find the average lengths of the opposite sides and multiply the averages together.

Guida de Abreu

C is false from the point of view of C' if it exists a function of representation ƒ: L→L', and it exists a problem in P which solution ζ it true following C but ƒ(ζ) is false following C'

"Generality" and "falsity" are not properties of conceptions but relations between conceptions whose validity depends on the translation from one system of representation to the other.

This is often hidden by the fact that we tend to read the production and the processes learners (or past mathematician) carry out directly in "our" mathematical terms.

Conception, knowing and concept

Are two conceptions referring to the same “object”?

Difficult question in mathematics where the only tangible things available are representations, but Vergnaud’s postulate (1981) offers a solution:

problems are sources and criteria of knowing

(fr. Connaissance)

Let C, C’ and C_a be three conceptions such that it exists functions of representation $f: L \rightarrow L_a$ and $f': L' \rightarrow L_a$

[*C and C’ have the same object with respect to C_a if for all p from P it exists p’ from P’ such that $f(p)=f'(p')$, and reciprocally*]

Conceptions have the same object if their spheres of practice can be matched from the point of view of a more general conception

→ *in practice, it is the conception of the researcher/teacher*

Tall's cognitive worlds

Tall three forms of cognitive development of mathematical objects

- An *embodied world* produced by the thinking about things that we perceive materially or mentally, supported *by the use of increasingly sophisticated language*
- A *proceptual world* made of *symbols* and *processes* on symbols expressing operations on more and more abstract mathematical concepts.
- A *formal world* based on object defined by properties, *expressed in terms of formal definitions* to form *the axiomatic formalism* needed to specify mathematical structures.

→ Working in a formal world gives rise to new material or mental realities, turning mathematical concepts into objects that can be manipulated and explored.

This transformation underpins a new embodied world within which will be born new concepts and new symbolisms.

In each world, the question of truth is addressed and solved by means that are appropriate to the available representations, ways of processing them, shaping justifications and "warrants for truth".

The tool-object dialectic

(Douady 1986)

→ Proof, an elusive concept

Source:
Balacheff, N. (preprint). Mathematical Argumentation, a Precursor Concept of Mathematical Proof. Proceedings ICME14 Invited Lectures, 17.
Mariotti, M. A. (2006). Proof and proving in mathematics education. In Á. Gutiérrez & P. Boero (Éds.), Handbook of Research on the Psychology of Mathematics Education (p. 173-204). Sense Publishers.
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Weber (2018) Understanding the Mathematical Way of Thinking – The Registers of Semiotic Representations. Springer Cham

Proof… What do you mean?

Institutional definition:

Proving is a competency (set of expected behaviors) in the Reasoning category:

making deductions based on specific hypotheses and rules, and justifying the results.

Empirical case studies of mathematicians norms or of their disagreements illustrate the limits of claiming unanimous consensus about what counts as proof (Weber):

- They are virtually unanimous about the validity of typical proofs (those of students)
- They can strongly disagree about atypical and innovative proofs

A discipline-based consensus:

A theorem is acceptable because it is systematised within a theory, with a complete autonomy from any verification or argumentation at an empirical level (Mariotti):

Theorem → (Theory, Sentence, Proof)

A human factor in mathematicians recognition of what counts as proof: belief and trust, understanding and insight, required level of rigor and explicitness, etc.

A cluster-definition of mathematical proof

A characterisation of proof which discriminates argumentations accepted as proof from those which are not

Czocher & Keith Weber

1. A proof is a convincing argumentation that will remove all doubt that a theorem is true for a knowledgeable mathematician.
2. A proof is a perspicuous argumentation (clear and precise) that is comprehensible by a knowledgeable mathematician and provides an understanding of why a theorem is true.
3. A proof is an a priori argumentation that shows that a theorem is a deductive consequence of axioms, assumptions and/or previously established claims. ← Deduction
4. A proof is a transparent argumentation where any sufficiently knowledgeable mathematician can fill in every gap [...] perhaps to the level of being a formal derivation.
5. A proof is an argumentation that has been sanctioned by the mathematical community. ← Authority

A cluster-definition of *mathematical argumentation*

For a characterization viable in the changing contexts of the mathematics curicula

Balacheff 2021

Mathematical argumentation is a multimodal text which is built in support to the truth of a sentence and contextualised by a state of knowledge

It requires each of the following components to be at least partly satisfied and sanctioned by the teacher:

- An *explicit* knowledge base established by and for the classroom community (including students and the teacher)
- A *linguistically appropriate* sentence, semantically adequate, of a general stance;
- A *structurally coherent* argumentation, ethically minded, congruent to students' conceptions, linking the sentence to the knowledge base.

Three regimes of controls

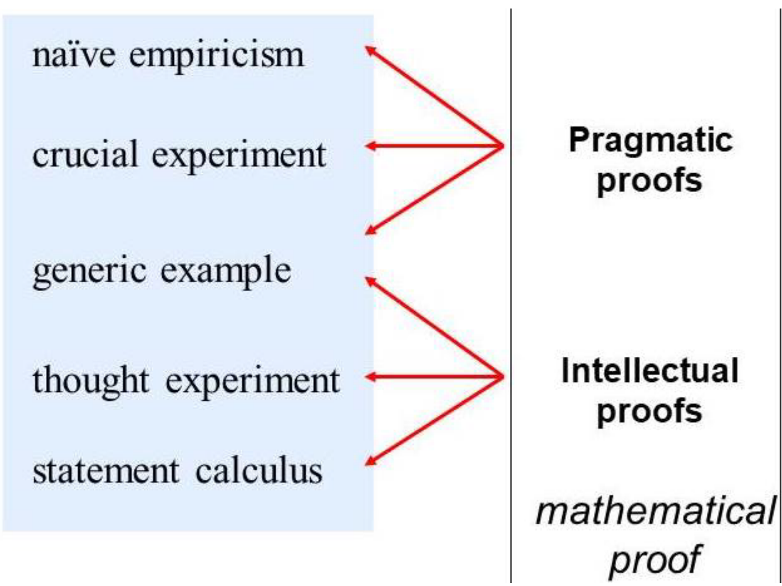
- In problem-solving, controls predicate upstream of a decision
- In the construction of an argumentation (discursive organisation, epistemic value)
- In a standardized argumentation in mathematics (discursive form, ontic reduction)

✓mathematical argumentation

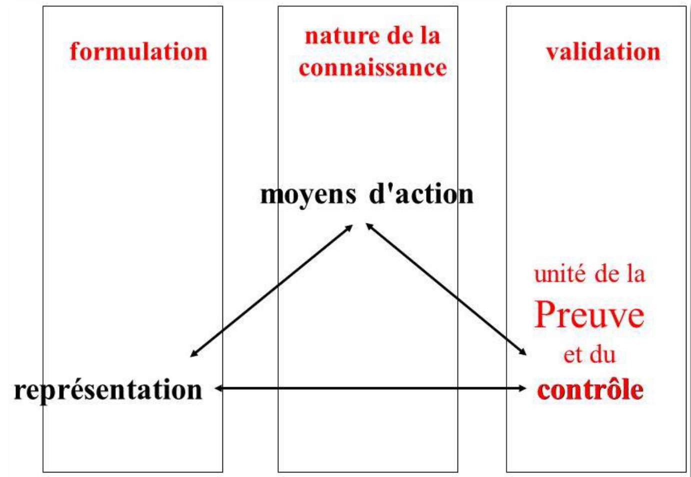
Weak discursive distance between argumentation and mathematical proof (Duval 1992)

✓institutionalisation of forms of proofs

Types of “argumentation” in the math classroom



The meaning of a validation process cannot be grasped without examining the conceptions that students mobilise, and the way they read the situation in which they find themselves.



A form of proof reflects
→ a principle of economy of logic.
→ the available **conceptions**

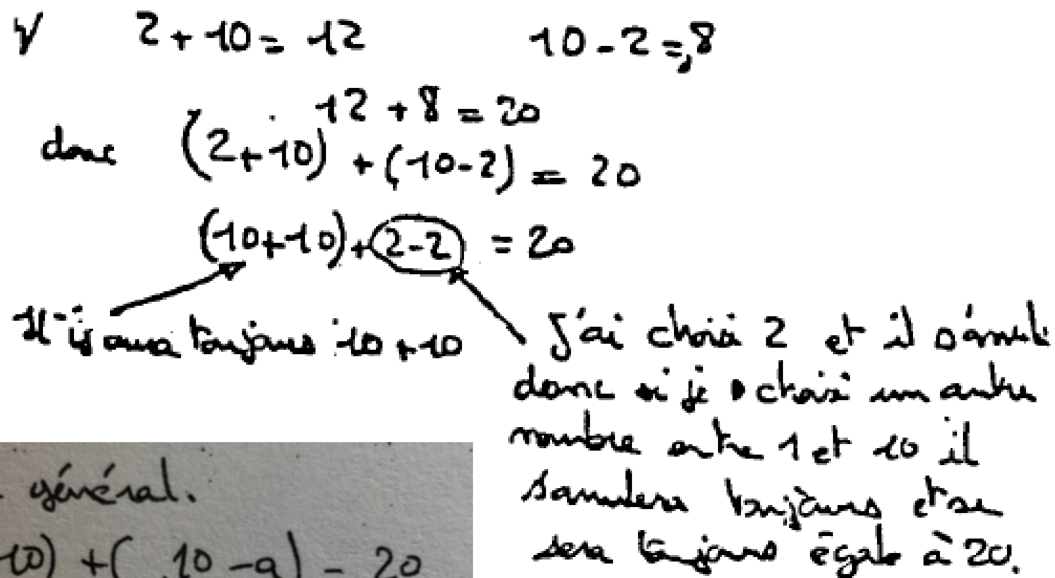
Balacheff 1987, 1988

Generic example

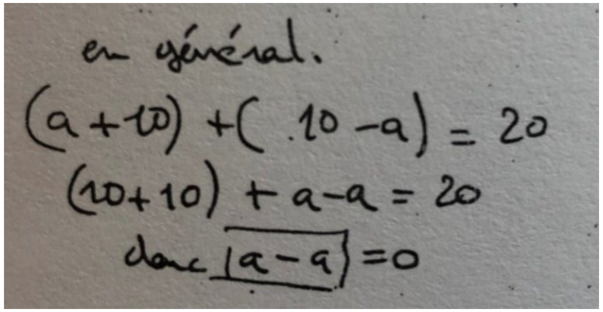
The challenge of representations

- → objects
- → relationships

Explanation of reasons



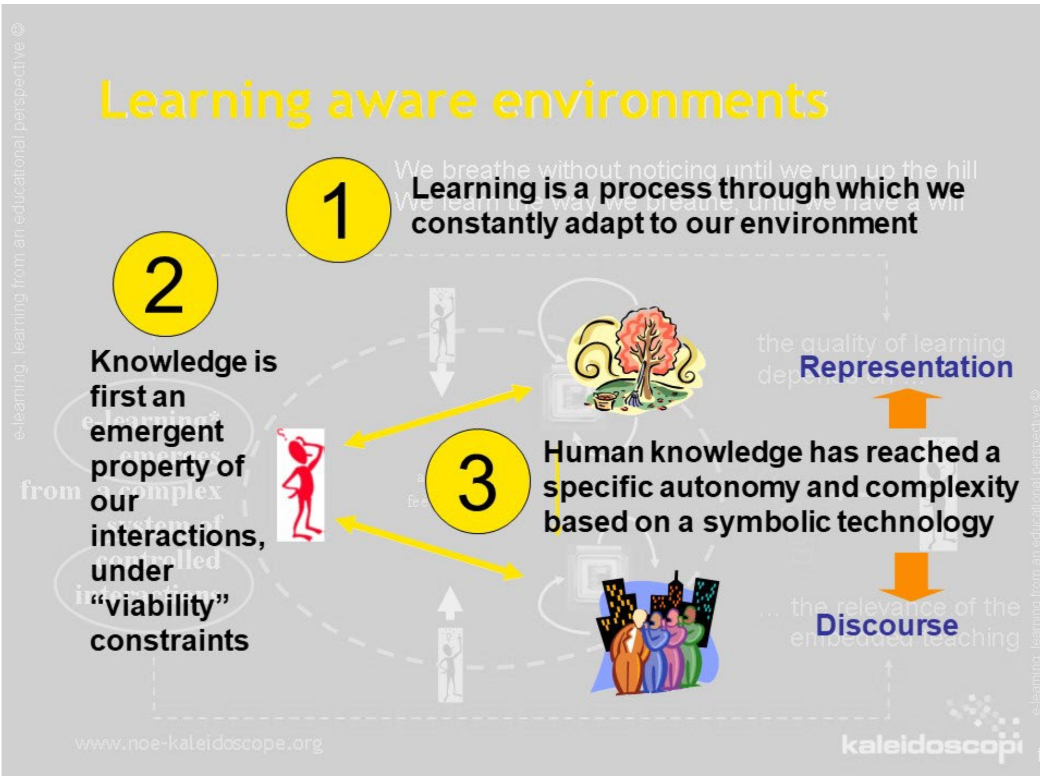
generalisation and construction of probationary nature

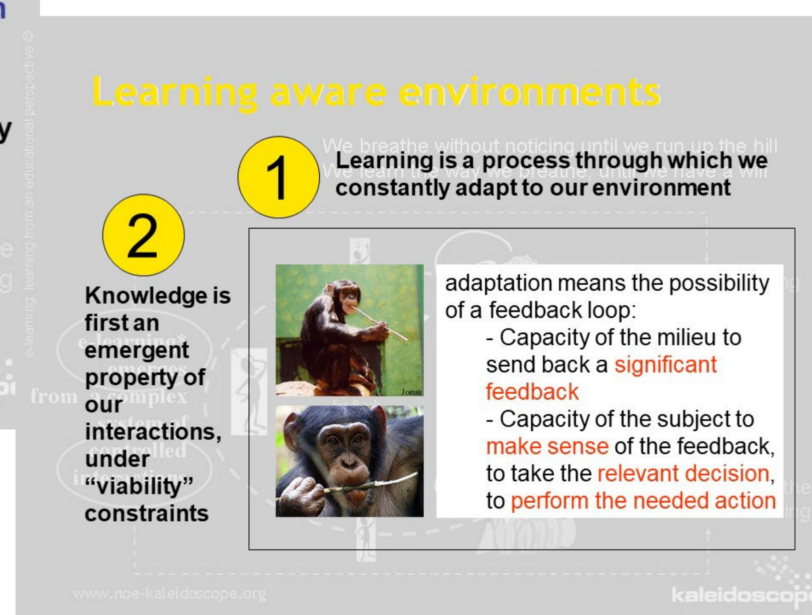


rethorics → heuristics

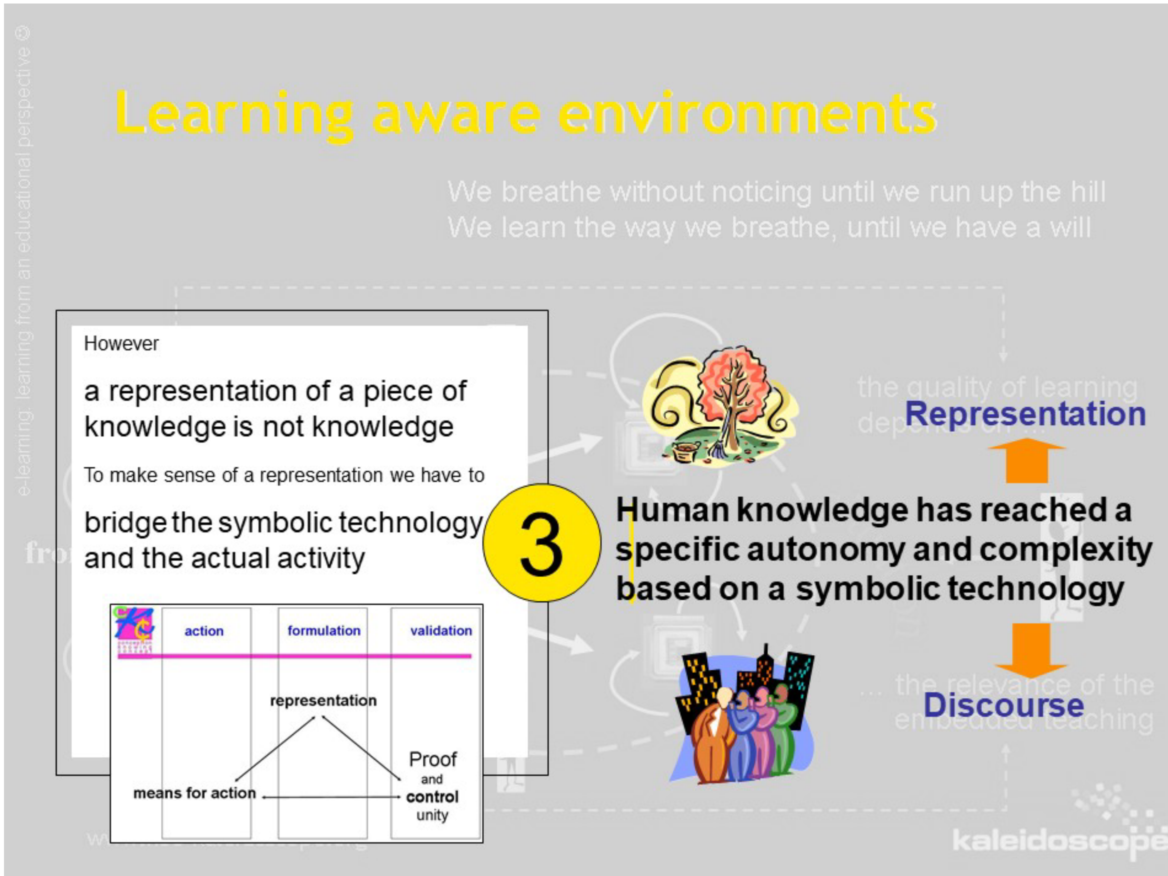
epistemic → ontic

→ Concluding comments





Which knowledge emerges from PATeaching: on PAT? on logic? on mathematics



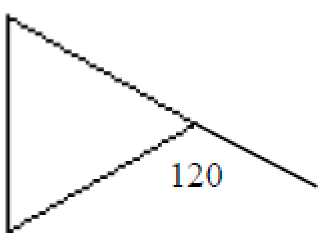
Learning aware environments
We breathe without noticing until we run up the hill
We learn the way we breathe, until we have a will
However
a representation of a piece of knowledge is not knowledge
To make sense of a representation we have to
bridge the symbolic technology and the actual activity
action
formulation
validation
representation
means for action
Proof
and
control
unity
3
Human knowledge has reached a specific autonomy and complexity based on a symbolic technology
Representation
Discourse
e-learning: learning from an educational perspective ©
kaleidoscope

Learning with a Proof Assistant

PAT characteristics

- The set of problems that the system can pose, in relation to the knowledge at stake in the learning process
- The actions that are possible with the available PAT tools
- The controls that the user can have on his decision and on managing feedback

Assess and manage the distance between the learner's conceptions and PAT conception, especially the semiotic distance



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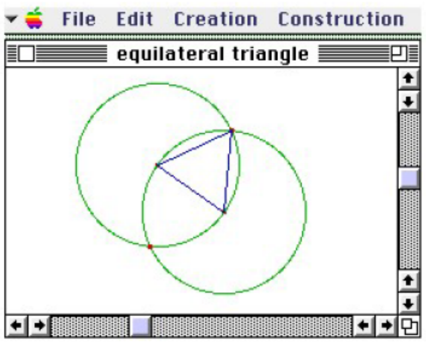
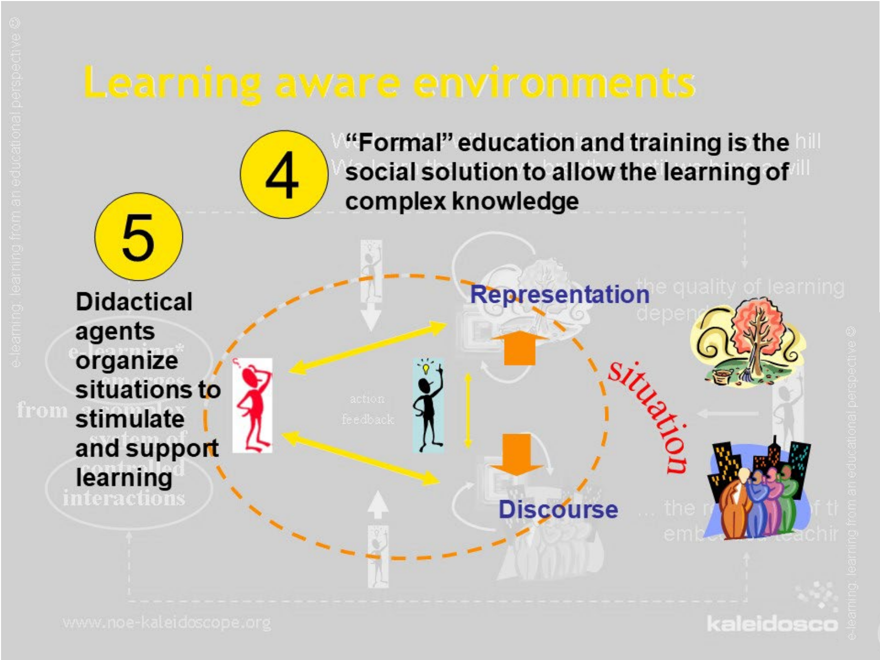
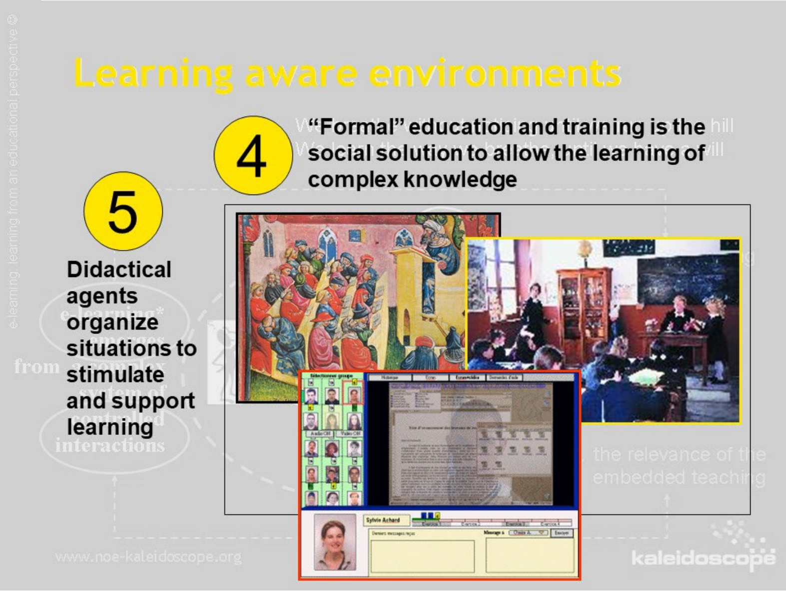


Teaching?


Learning aware environments
4
"Formal" education and training is the social solution to allow the learning of complex knowledge
5
Didactical agents organize situations to stimulate and support learning
Representation
situation
Discourse
www.noe-kaleidoscope.org
kaleidosco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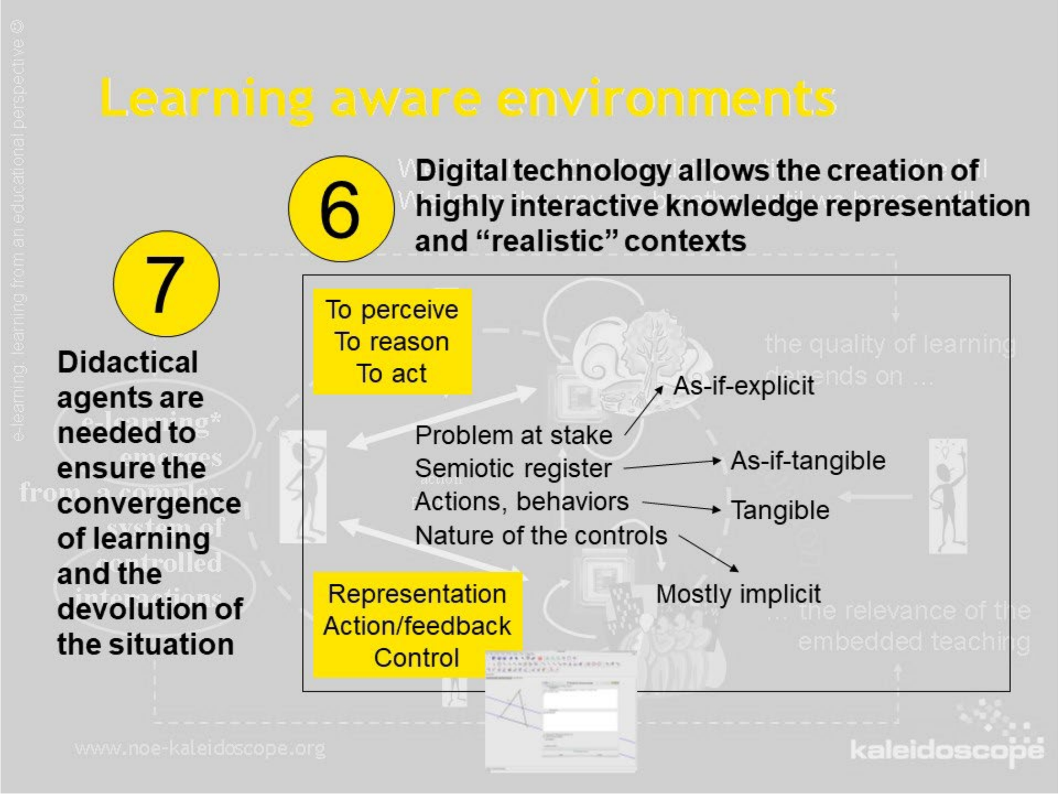
PATeaching



→ What makes the problem problematic: the situation

Problematisation of knowledge

The question then is: which services can PAT provide?

What kind of feedback while the student is engaged in problem solving?

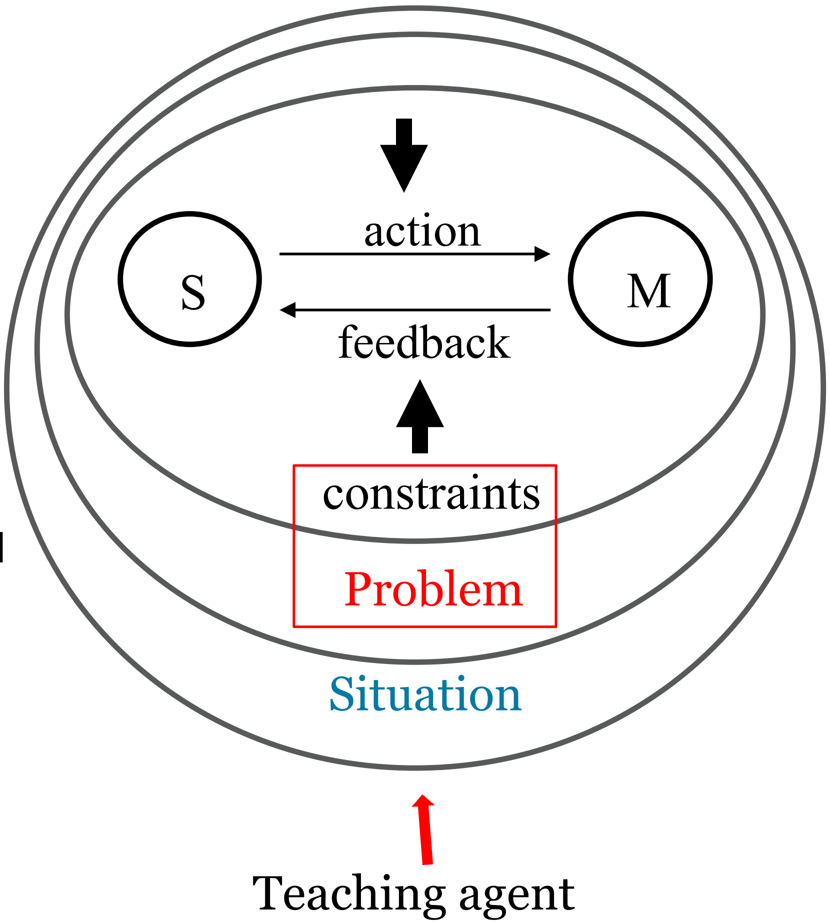
It could target

- the strategy rather than a particular rule,
- the related knowledge or the logical sequence,

It could

- immediately spot the misuse of a theorem, a flaw in reasoning,
- wait to provide a counterexample to the proposed proof.

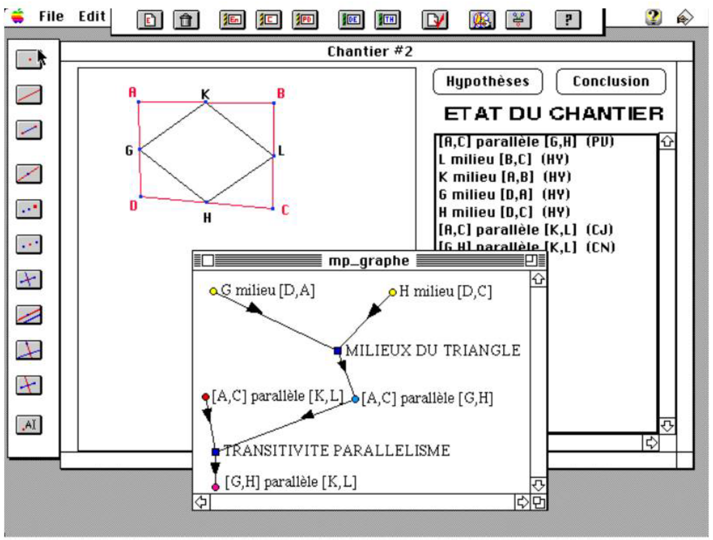
It could be textual and/or visual and, in the case of geometry

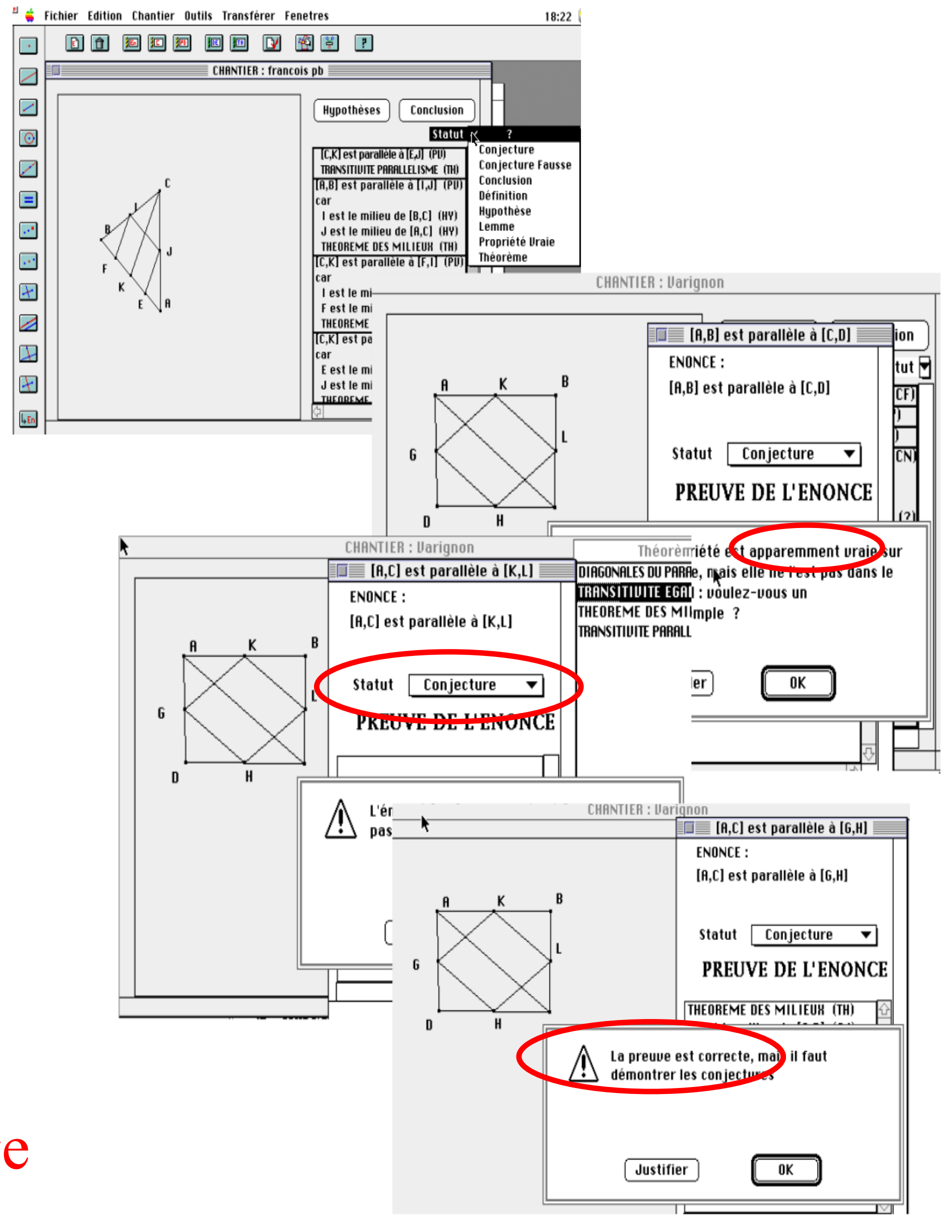


PAT is the support of a milieu
it has the behaviour of an agent

Cabri-Euclide

- Multiple, linked representation registers
- deductive form, a structuring tool
- Epistemic value of statements

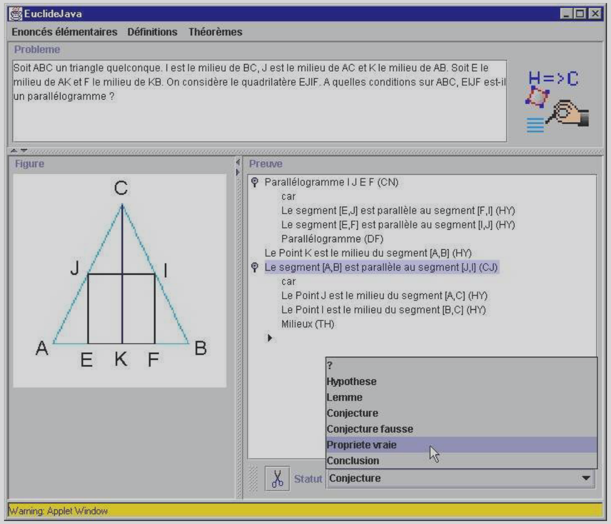




Analyse and compare proofs as objects

Proof are organisers of knowledge

Baghera



↔ ATINF

(Inference workshop, RicardoCaferra)

- a logical and algebraic approach to the automatic proof of theorems & …
- process automatic proofs to give an intelligible explanation

Manipulation of proof bases and counter-examples, discovery of analogies, proof plans

↕ student

Didactical situation

Teaching agentS

Very last message…

Along with providing features and functionality for mathematicians' activities PATutors must take three additional categories of users into account:

- the curriculum decision makers (who specify the standard of mathematical validation at a given grade),
- the teachers (who orchestrate learning and decide what counts as a proof in relation to a standard),
- the learners (who are simultaneously constructing an understanding of proof and of the related content).

Thank you!

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